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Final Technical Report

by
Prof.Dr.F.Kappel

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Research Center, Graz (Austria)
A-8010 Graz, Steyrergasse 17
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20. ABSTRACT

This report is in two parts. In the first part averaging projections on finite dimensional subspaces are used in order to approximate an autonomous nonlinear functional differential equation with state space $\mathbb{R}^n x L^p [-r,0;\mathbb{R}^n)$ by a sequence of ordinary differential equations. This type of approximations already proved to be successful for the numerical treatment of hereditary control problems involving linear functional differential equations.

In the second part the authors investigate the optimal control problem for Volterra integrodifferential equations.

$$X(T) = A(t)x(t) + \int_{-\infty}^{t} F(t-s)x(s)ds + B(t)u(t)$$

where the target sets are elements of some function space,

$$X(t) = \varphi(t) \quad \text{for} \quad 0 < T-r \leq t \leq T$$

The approach used is the abstract theory of Dubovitskii-Milyutin, the result is a necessary condition in form of a maximum principal.

A. Projection Methods in Nonlinear Autonomous Functional Differential Equations.

Abstract

In this part averaging projections on finite dimensional subspaces are used in order to approximate an autonomous nonlinear functional differential equation with state space $\mathbb{R}^n \times L^p[-r,0;\mathbb{R}^n)$ by a sequence of ordinary differential equations. This type of approximations already proved to be successful for the numerical treatment of hereditary control problems involving linear functional differential equations.

Contents

& 1. Introduction

For the numerical treatment of hereditary control problems various authors have develloped an approach which uses projections on finite dimensional subspaces of the given state space (see especially [4] and [2] and the extension discussion on related literature given in [2]. Problems considered are of the following type:

(P) Minimize $\Phi(u) = J(x(\cdot,u),u)$ subject to

$$\dot{x}(t,u) = f(x(t,u),x_t(u),u(t)),\ t\in[0,t_1],$$
$$x(0,u) = \eta\in\mathbb{R}^n,$$
$$x(s,u) = \varphi(s) \text{ a.e. on } [-r,0),\ \varphi\in L^p(-r,0;\mathbb{R}^n),$$

and

$$u\in U,\ U \text{ a closed convex subset of } L^2([0,t_1],\mathbb{R}^m).$$

For any function $x:[-r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, the symbol x_t denotes the map $s \to x(t+s)$, $s\in[-r,0]$, $t \geq 0$. The state space used in (P) is $X = \mathbb{R}^n x L^p(-r,0;\mathbb{R}^n)$ and is therefore infinite dimensional.
A possible approach to problem (P) can be described as follows. Choose a sequence (X^N) of finite dimensional subspace of X such that $X^N \subset X^{N+1}$ for all N with projection $\pi^N:X \to X^N$, $\pi^N \to id_X$ as $N \to \infty$. Define operators G^N such that the solution of the functional differential equation in (P) is approximated by the sequence $(x^N(t;u))$ of solutions of

$$\dot{x}^N(t) = G^N(x^N(t),u(t)) \ ,\ t \geq 0,$$
$$x^N(0;u) = \pi^N(\eta,\varphi).$$

Moreover, we have to define a sequence $\Phi^N(u) = J^N(x^N(\cdot,u),u)$ of cost functionals.

Therefore in addition to (P) we have a sequence of problems:

(P^N) Minimize $\Phi^N(u)$ subject to

$$\dot{x}^N(t;u) = G^N(x^N(t;u), u(t)) , t\in[0,t_1],$$
$$x^N(0;u) = \pi^N(\eta,\varphi),$$
and
$u\in U$.

Suppose that (P) and (P^N) have the solutions $\bar{u},\bar{u}^N$, respectively. Then the question is: Under what condition we have $\bar{u}^N \to \bar{u}$ (weakly or strongly), $x^N(t;\bar{u}^N) \to x(t;\bar{u})$ and $\Phi^N(\bar{u}^N) \to \Phi(\bar{u})$

If convergence can be prooved then we have an approximation of the infinite dimensional problem (P) by the sequence of finite dimensional problems (P^N).

The approach indicated here was applied successfully for the numerical solution of problems (P) where the functional differential equation is linear (cf. [2], [3]). The following sections deal with the approximation problem for the functional-differential equation only assuming local Lipschitz type conditions. The results presented here constitute the first but essential step towards the numerical treatment of nonlinear hereditary control problems by projection methods.

As mentioned above an important feature of this investigation is that only local Lipschitz type conditions are assumed. This implies that in general we only have local existence of solutions. But even in the case where we have global existence of solutions, the solution semigroup in general is not a semigroup with dissipative infinitesimal generator. This is only the case in special situations [20]. Therefore the results of the existing theory of nonlinear semigroups cannot be applied directly.

In Section 2 we provide the existence, uniqueness and continuous dependence results which we need in this paper. Section 3 gives an analysis of the local semigroup generated by solutions of the underlying functional-differential equation. The main idea

in Section 4 is to approximate the solution semigroup by semigroups with nicer properties (i.e. dissipative infinitesimal generator). These approximating semigroups are solution semigroups of "averaged" functional differential equations. In Section 5 the results of Section 4 are carried over to the local case. Finally, in Section 6 the averaging projections introduced in this connexion by Banks and Burns [2] (see also the extensive literature given there) are considered in the case of difference-differential equations where the right-hand side is locally Lipschitz. For Lipschitz continuous initial data we get uniform convergence of the approximating ordinary differential equation solutionson compact intervals.

The methods applied in the following sections will be carried over to time-dependent equations and equations with infinite delay.

& 2. Preliminaries

We denote with $\mathcal{L}^p(a,b;\mathbb{R}^n)$, $1 \leq p < \infty$, $-\infty < a < b < \infty$, the linear space of functions $x:[a,b] \to \mathbb{R}^n$ such that $|x|^p$ is integrable on $[a,b]$. $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. $L^p(a,b;\mathbb{R}^n)$ is the Banach-space of equivalence classes of functions in $\mathcal{L}^p(a,b;\mathbb{R}^n)$ with norm $\|\cdot\|_p$.
For notational purposes we also introduce the spaces $L^{p,loc} = L^{p,loc}(0,\infty;\mathbb{R}^n)$ of equivalence classes of functions $x:[0,\infty) \to \mathbb{R}^n$ such that $|x|^p$ is locally integrable, $1 \leq p < \infty$. A family of seminorms on $L^{p,loc}$ is given by $\|x\|_{p,T} = (\int_0^T |x|^p \, dt)^{1/p}$, $T > 0$. $C(a,b;\mathbb{R}^n)$ will be the Banach-space of continuous functions $x:[a,b] \to \mathbb{R}^n$, $-\infty < a < b < \infty$, with norm $\|x\|_C = \sup_{[a,b]} |x|$.
It will not be necessary to indicate the interval $[a,b]$ in the various norms.

If x is a function $[-r,\alpha) \to \mathbb{R}^n$, $\alpha > 0$, $0 < r < \infty$, then x_t, $0 \leq t < \alpha$, denotes the function $[-r,0] \to \mathbb{R}^n$ defined by $x_t(s) = x(t+s)$, $s \in [-r,0]$. $\mathcal{X}$ will be the linear space $\mathbb{R}^n \times \mathcal{L}^p(-r,0;\mathbb{R}^n)$, $1 \leq p < \infty$, and X the Banach-space $\mathbb{R}^n \times L^p(-r,0;\mathbb{R}^n)$ with norm $\|(\vartheta,\psi)\| = (|\vartheta|^p + \|\psi\|_p^p)^{1/p}$. In this section we shall use the equivalent norm $\|(\vartheta,\psi)\|_0 = |\vartheta| + \|\psi\|_p$ which simplifies the estimates to be made in the proofs. $\mathcal{C}$ denotes the set $\{(\varphi(0),\varphi) \mid \varphi \in C(-r,0;\mathbb{R}^n)\}$.

Let f be a map $\mathcal{D}(f) \to \mathbb{R}^n$ where $\mathcal{C} \subset \mathcal{D}(f) \subset \mathcal{X}$ and take $g \in L^{p,loc}$. Given any $\tilde{x}_0 = (\eta,\varphi) \in \mathcal{X}$ we consider the Cauchy problem

$$\dot{x}(t) = f(x(t),x_t) + g(t), \quad t \geq 0, \tag{1.1}$$

$$x(0) = \eta, \quad x(s) = \varphi(s) \quad \text{a.e. on } [-r,0). \tag{1.2}$$

A solution of this problem is denoted by $x(t) = x(t;\tilde{x}_0,g)$ or

$x(t) = x(t;\eta,\varphi,g)$, respectively, and is a function $[-r,\infty) \to \mathbb{R}^n$, $\alpha > 0$, such that (1.2) holds, x is absolutely continuous on $[0,\alpha)$ and (1.1) is true a.e. on $[0,\alpha)$.

In order to get existence, uniqueness and continuous dependence of solutions we impose the following conditions on f:

(H1) If, forsome $\alpha > 0$, $x \in L^p(-r,\alpha;\mathbb{R}^n)$ and x is continuous on $[0,\alpha]$ then the map $t \to f(x(t),x_t)$ is defined a.e. on $[0,\alpha]$, depends on the equivalence class of x only and is in $L^1(0,\alpha;\mathbb{R}^n)$.

(H2) Given $\beta > 0$, there exists a continuous nonnegative and nondecreasing function $\gamma(t) = \gamma_\beta(t)$ defined on $[0,\infty)$ such that for any $\alpha > 0$ the inequality
$$\int_0^t |f(x(s),x_s) - f(y(s),y_s)|ds \leq \gamma(t)\left(\int_{-r}^t |x(s)-y(s)|^p ds\right)^{1/p}$$
is satisfied for all $t \in [0,\alpha]$ and all functions x,y in $L^p(-r,\alpha;\mathbb{R}^n)$ which are absolutely continuous on $[0,\alpha]$ with $\|x\|_p \leq \beta$, $\|y\|_p \leq \beta$.

Remark 1.1. Since equation (1.1) should incorporate difference-differential equations, it is not reasonable to demand that $\mathcal{D}(f) = \mathcal{X}$. To be more precise, condition (H1) for instance should read as follows: Given $x \in L^p(-r,\alpha;\mathbb{R}^n)$ such that there exists a representation of x which is continuous on $[0,\alpha]$ then for each representation $\hat{x}$ of x the map $t \to f(\hat{x}(t),\hat{x}_t)$ is in the same equivalence class of $L^1(0,\alpha;\mathbb{R}^n)$. An analogous statement is appropriate with respect to condition (H2). But, since expressions like $f(x(s),x_s)$ will appear only under the integral sign, the formulation of conditions (H1) and (H2) as given above will not cause any confusion. However, in Section 2 we shall need the following definition of $f(\eta,\varphi)$ for those $(\eta,\varphi) \in X$ such that there is a representation $\hat{\varphi}$ of φ with $\hat{\varphi}(0) = \eta$ which is continuous on $[-r,0]$. In this case

$f(\eta,\varphi) = f(\hat{\varphi}(0),\hat{\varphi})$ defines $f(\eta,\varphi)$ uniquely, because $\hat{\varphi}$ is uniquely determined.

Remark 1.2. Let equation (1.1) be a difference-differential equation

$$\dot{x}(t) = h(x(t),\ x(t-k_1),\dots,x(t-k_m)), \tag{1.3}$$

$k_o=0 < k_1 < \dots < k_m$, where h is defined on $\mathbb{R}^{n(m+1)}$ and Lipschitz-continuous, i.e. for each $M > 0$ there exists a constant $L(M) > 0$ such that

$$|h(y_o,\dots,y_m)-h(z_o,\dots,z_m)| \le L(M)\sum_{j=o}^{m}|y_j-z_j|$$

for all $(y_o,\dots,y_m)$ and $(z_o,\dots,z_m)$ such that $\sum_{j=o}^{m}|y_j| \le M$ and $\sum_{j=o}^{m}|z_j| \le M$. Suppose there exist numbers

$\rho \ge 0$, $\sigma \ge 0$, and $q \ge 0$ such that

$$L(M) \le \sigma M^q + \rho \tag{1.4}$$

for $M \ge 0$. Then

$$J(t) = \int_o^t|f(x(s),x_s) - f(y(s),y_s)|ds$$
$$\le \sum_{j=o}^{m}\int_o^t(\sigma\delta^q(s)+\rho)|x(s-k_j) - y(s-k_j)|ds,$$

where $\delta(s)=\max(\sum_{j=o}^{m}|x(s-k_j)|,\ \sum_{j=o}^{m}|y(s-k_j)|) \le \sum_{j=o}^{m}|x(s-k_j)| +$ $+\sum_{j=o}^{m}|y(s-k_j)|$. Since $\delta\in L^p(0,t;\mathbb{R})$ we have $\delta^q(s)\in L^{1-1/p}(0,t;\mathbb{R})$ provided that

$$q \le p - 1 .$$

Let us take $p = q+1$. Then an application of the Hölder inequality gives

$$J(t) \le \{\rho t^{1-1/p} + \sigma(\int_o^t \delta^p(s)ds)^{1-1/p}\}\sum_{j=o}^{m}(\int_o^t|x(s-k_j)-y(s-k_j)|^p ds)^{1/p}$$

$$\leq (m+1)\{\rho t^{1-1/p} + \sigma (\int_0^t \delta^p(s)ds)^{1-1/p}\} \ (\int_{-r}^t |x(s) - y(s)|^p ds)^{1/p} .$$

From Minkowski's inequality we get

$$(\int_0^t \delta^p(s)ds)^{1/p} \leq \sum_{j=0}^m (\int_0^t |x(s-k_j)|^p ds)^{1/p} + $$
$$+ \sum_{j=0}^m (\int_0^t |y(s-k_j)|^p)^{1/p} \leq 2(m+1)\beta$$

i.e. we have (H2) where $\gamma_\beta(t)$ is given by

$$\gamma_\beta(t) = (m+1)(\rho t^{1-1/p} + \sigma(2(m+1)\beta)^{p-1}).$$

It is easy to see that also condition (H1) holds.

Remark 1.3. Another case covered by conditions (H1) and (H2) is where f is a Lipschitz-continuous map $L^p(-r,0;\mathbb{R}^n) \to \mathbb{R}^n$, i.e.

$$|f(\varphi) - f(\psi)| \leq L(M)\|\varphi-\psi\|_p \tag{1.5}$$

where $M = \max(\|\varphi\|_p, \|\psi\|_p)$. Then an easy calculation shows that (H2) holds with $\gamma_\beta(t) = tL(\beta)$.

Remark 1.4. Conditions (H1) and (H2) are the local versions of global conditions already used by Borisovič and Turbabin [6] and Banks and Burns [2] in the linear case. The conditions apply to functions which are candidates for solutions of Cauchy problems and give restrictions for the right-hand side of equation (1.1) along absolutely continuous prolongations of initial data. For existence and uniqueness results it is this behaviour which is essential. Related existence results were obtained in [12] and [13].

Proposition 1.1. Suppose that (H1) and (H2) hold for f and take $\delta > 0$. Then there exists a constant $\alpha = \alpha(\delta) > 0$ such that the Cauchy problem (1.1), (1.2) has a unique solution $x(t;\tilde{x},g)$ existing on $[-r,\alpha]$ provided that $\|\tilde{x}\|_0 \leq \delta$ and $\|g\|_{p,\alpha} \leq \delta$.

Moreover, we have $|x(t)| \le \delta+1$ on $[0,\alpha]$.

Remark 1.5. The proof of this proposition will show that solutions of (1.1), (1.2) are dependent on the class of $\tilde{x}$ only. Therefore, it is legitimate to consider initial data in X.

Proof of Proposition 1.1. Take $\alpha_0 > 0$ and define for $\tilde{x} = (\eta,\varphi)\in X$ with $\|\tilde{x}\|_0 \le \delta$ the function

$$\tilde{\eta}(s) = \begin{cases} \varphi(s) & \text{for } s\in[-r,0), \\ \eta & \text{for } s\in[0,\alpha_0]. \end{cases}$$

We further define

$$S = \{\psi\in C(0,\alpha_0;\mathbb{R}^n) \mid \|\psi-\eta\|_C \le 1\},$$

where η also denotes the constant function $\equiv\eta$ on $[0,\alpha_0]$, and for $\psi\in S$ the function

$$\tilde{\psi}(s) = \begin{cases} \varphi(s) & \text{for } s\in[-r,0), \\ \psi(s) & \text{for } s\in[0,\alpha_0]. \end{cases}$$

The operator T is defined by

$$(T\psi)(t) = \eta+\int_0^t f(\psi(s),\tilde{\psi}_s)ds + \int_0^t g(s)ds, \tag{1.6}$$

$t\in[0,\alpha_0]$, $\psi\in S$. Note, that $\tilde{\psi}_s\in L^p(-r,0;\mathbb{R}^n)$ for $s\in[0,\alpha_0]$. Therefore, according to (H1), $T\psi$ exists and is absolutely continuous on $[0,\alpha_0]$. For $\psi\in S$ we have $\|\tilde{\psi}\|_p \le (\delta+1)(1+\alpha_0)^{1/p}$. We choose $\gamma(t)$ according to (H2) for $\beta=(\delta+1)(1+\alpha_0)^{1/p}$. Then for $\alpha_1\in(0,\alpha_0]$ and $\psi,\chi\in S$, we get the estimates

$$|(T\psi)(t)-\eta| \le \gamma(\alpha_1)\left(\int_{-r}^{\alpha_1} |\tilde{\psi}(s)|^p ds\right)^{1/p} +$$

$$+ \int_0^{\alpha_1} |f(0,0)|ds + \int_0^{\alpha_1} |g(s)|ds$$

$$\le \gamma(\alpha_1)\beta + \alpha_1|f(0,0)| + \int_0^{\alpha_1} |g(s)|ds = \rho_1(\alpha_1)$$

and

$$|(T\psi)(t) - (T\chi)(t)| \le \left(\int_0^{\alpha_1} |\psi(s)-\chi(s)|^p ds\right)^{1/p} \gamma(\alpha_1)$$

$$\le \alpha_1^{1/p}\ \gamma(\alpha_1) \max_{[0,\alpha_1]} |\psi(s)-\chi(s)|$$

$$= \rho_2(\alpha_1) \max_{[0,\alpha_1]} |\psi(s)-\chi(s)|$$

for all $t\in[0,\alpha_1]$. Since $\lim_{\alpha\to 0+} \rho_i(\alpha) = 0$, $i = 1,2$, these estimates show that, for α_1 sufficiently small, (1.6) defines a contraction $S_0 \to S_0$, S_0 being the complete metric space

$$S_0 = \{\psi\in C(0,\alpha_1;\mathbb{R}^n) \mid \|\psi-\eta\|_C \le 1\}.$$

Therefore, T has a unique fixed point in S_0 which gives the unique solution to the Cauchy problem (1.1), (1.2) with $\|\tilde{x}\|_0 \le \delta$. Existence of these solutions is guaranteed on each interval $[0,\alpha_1]$ where α_1 is determined by the conditions $\rho_1(\alpha_1) \le 1$ and $\rho_2(\alpha_1) < 1$. By definition of S_0 it is clear that $|x(t;\tilde{x},g)| \le \delta+1$ on $[0,\alpha_1]$.

Remark 1.6. The proof of Proposition 1.1 shows that in order to get existence and uniqueness of solutions to the Cauchy problem (1.1), (1.2) condition (H2) can be weakened. The estimate in (H2) only needs to hold for those functions x,y which satisfy in addition $x(t)=y(t)$, $t\in[-r,0]$. For instance this weaker version of (H2) holds for the scalar equation

$$\dot{x}(t) = \operatorname{sgn} x(t-1)$$

whereas (H2) does not hold. The full power of condition (H2) is needed to prove continuous dependence of solutions on initial data. It is easily seen that in case of the example given above solutions do not depend continuously on inital data.

Remark 1.7. If condition (H2) is a global condition, i.e. the function $\gamma(t)$ does not depend on β, then solutions of (1.1), (1.2) exist globally. This is true because we can take $S = C(0,\alpha_0;\mathbb{R}^n)$ and $\rho_2(\alpha)$ is only dependent on α and $\gamma(t)$. This implies that each solution can be prolongated on an interval of length α_1 where α_1 is determined by $\rho_2(\alpha_1) < 1$.

Proposition 1.2. Suppose that conditions (H1) and (H2) hold.

a) If $x(t) = x(t;\eta,\varphi,g)$ and $y(t) = x(t;\tilde{\eta},\tilde{\varphi},\tilde{g})$ are two solutions existing on the same interval $[-r,T]$ then there exist nonnegative constants κ_0,κ_1 and σ such that

$$|x(t)-y(t)| \le \kappa_0 \|(\eta,\varphi)-(\tilde{\eta},\tilde{\varphi})\|_0 e^{\sigma t} + \kappa_1 \|g-\tilde{g}\|_{p,T} e^{\sigma t} \tag{1.7}$$

for $t\in[0,T]$. The constants are dependent on T, $\|x(\cdot)\|_p$, and $\|y(\cdot)\|_p$.

b) Let $x(t;\eta,\varphi,g)$ be a solution with maximal interval of existence $[-r,t^+)$, $0<t^+ \le \infty$. Then for every $T\in(0,t^+)$ there exists a positive constant τ_0 such that $x(t;\tilde{\eta},\tilde{\varphi},\tilde{g})$ exists on $[-r,T]$ provided that $\|(\tilde{\eta},\tilde{\varphi})-(\eta,\varphi)\|_0 + \|\tilde{g}-g\|_{p,T} < \tau_0$. The constant τ_0 is dependent on $\sup_{[0,T]} \|(x(t),x_t)\|_0$.

c) Let $x(t;\eta,\varphi,g)$ be as in b) and suppose that (η_n,φ_n) and (g_n) are sequences in X and $L^{p,loc}$ such that $(\eta_n,\varphi_n) \to (\eta,\varphi)$ and $\|g_n-g\|_{p,T} \to 0$ for each $T > 0$. Then

$$\lim_{n\to\infty} x(t;\eta_n,\varphi_n,g_n) = x(t;\eta,\varphi,g)$$

uniformly on compact subintervals of $[0,t^+)$.

Proof. We choose $\beta > 0$ such that $\|x\|_p \le \beta$ and $\|y\|_p \le \beta$.

Then with $\gamma(t) = \gamma_\beta(t)$ according to (H2) we obtain

$$|x(t) - y(t)| \leq |\eta-\tilde{\eta}| + \gamma(T)\|\varphi-\tilde{\varphi}\|_p + \gamma(T)\left(\int_0^t |x(s)-y(s)|^p ds\right)^{1/p}$$

$$+ T^{1-1/p}\|g-\tilde{g}\|_{p,T}.$$

If we define $v(t) = |x(t)-y(t)|$,
$V(t) = \int_0^t v^p(s)ds$ for $t \geq 0$ and $\rho = |\eta-\tilde{\eta}| + \gamma(T)\ \|\varphi-\tilde{\varphi}\|_p$
$+ T^{1-1/p}\ \|g-\tilde{g}\|_{p,T}$ we get

$$v(t) \leq \rho + \gamma(T)\ (V(t))^{1/p}\ ,\ t\in[0,T],$$

and

$$\dot{V}(t) = v^p(t) \leq (\rho+\gamma(T)(V(t))^{1/p})^p,\ t\in[0,T]. \qquad (1.8)$$

In case p=1 Gronwall's inequality gives the desired result. In the general case we have to proceed as follows. Substitute $y(t)=\gamma(T)V(t)^{1/p}$. Then integrating (1.8) we obtain

$$\frac{p}{\gamma(T)^p}\int_0^{\gamma_T V^{1/p}} \frac{(y+\rho-\rho)^{p-1}}{(\rho+y)^p}dy = t\ ,\ t\in[0,T].$$

If we note that $\dfrac{(y+\rho-\rho)^{p-1}}{(\rho+y)^p} = \sum\limits_{\nu=0}^{\infty}\binom{p-1}{\nu}\dfrac{(-1)^\nu\rho^\nu}{(y+\rho)^{\nu+1}}$

for nonnegative y then termwise integration gives

$$\ln(\rho+\gamma(T)V^{1/p}) + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu}\binom{p-1}{\nu}\left(\frac{\rho}{\rho+\gamma_T V^{1/p}}\right)^\nu$$

$$\leq (\gamma(T)^p/p)t + \ln\rho + \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu+1}}{\nu}\binom{p-1}{\nu}.$$

From this we get the estimate

$$\ln(\rho+\gamma(T)V^{1/p}) \leq (\gamma(T)^p/p)t + \ln\rho + 2\sum_{k=0}^{\infty}\frac{1}{2k+1}\binom{p-1}{2k+1}$$

or

$$v(t) \leq \rho+\gamma(T)V^{1/p} \leq \rho\tilde{\kappa}\exp(\gamma(T)^p/p)t\ ,t\in[0,T],$$

where $\tilde{\kappa} = \exp(2 \sum_{k=0} \frac{1}{2k+1}\binom{p-1}{2k+1})$. Note, that the series appearing in this proof are convergent. According to the definition of ρ and $v(t)$ the last estimate is just (1.7) with $\sigma = \gamma(T)^p/p$, $\kappa_0 = \tilde{\kappa} \max(1, \gamma(T))$ and $\kappa_1 = \tilde{\kappa}\, T^{1-1/p}$.

In order to prove part b) we choose $\delta > 0$ such that $\|(x(t), x_t)\|_0 < \delta$ on $[0,T]$ and $\|g\|_{p,T} < \delta$. Then according to Proposition 1.1 there exists a constant $\alpha > 0$ dependent on δ only such that each solution $y(t) = x(t;\tilde{\eta},\tilde{\varphi},\tilde{g})$ exists on $[-r,\alpha]$ provided that $\|(\tilde{\eta},\tilde{\varphi})\|_0 \leq \delta$ and $\|\tilde{g}\|_{p,\alpha} \leq \|\tilde{g}\|_{p,T} \leq \delta$. Moreover $|y(t)| \leq \delta+1$ on $[0,\alpha]$. Then by part a) of Proposition 1.2 there exists a constant $\tau_1 > 0$ (dependent on δ only) such that

$$\|(\eta,\varphi) - (\tilde{\eta},\tilde{\varphi})\|_0 + \|g-\tilde{g}\|_{p,T} < \tau_1 \tag{1.9}$$

implies $\|(y(\alpha), y_\alpha)\|_0 < \delta$. Then by Proposition 1.1 all solutions $y(t)$ with (1.9) exist on $[-r,2\alpha]$ with $|y(t)| < \delta+1$ on $[0,2\alpha]$. By part a) of Proposition 1.2 we can choose a $\tau_2 \leq \tau_1$ such that $\|(y(2\alpha), y_{2\alpha})\|_0 < \delta$ for all solutions $y(t)$ such that (1.9) holds with τ_1 replaced by τ_2. Proceeding in this way we arrive in a finite number of steps at a number $\tau_0 > 0$ which has the properties stated in part b) of the proposition.

Part c) is an easy consequence of parts a) and b).

Remark 1.8. If condition (H2) is a global condition then an inspection of the proof given for part a) shows that the constants σ, κ_0 and κ_1 are independent of $x(\cdot)$ and $y(\cdot)$.

§ 3. The nonlinear local semigroup

In this section we consider the autonomous equation

$$\dot{x}(t) = f(x(t),x_t),\ t \geq 0. \tag{2.1}$$

Conditions (H1) and (H2) are assumed to hold for f throughout this section. Given $\tilde{x}_0 = (\eta,\varphi)\in X$ we simply write $x(t;\eta,\varphi)$ or $x(t;,\tilde{x}_0)$ instead of $x(t;\eta,\varphi,0)$ or $x(t;\tilde{x}_0,0)$, respectively.

To each $\tilde{x}\in X$ there corresponds the maximal interval of existence $[-r,t^+_{\tilde{x}})$ for $x(t;\tilde{x})$ where $0<t^+_{\tilde{x}} \leq \infty$. It is clear that $(x(t;\tilde{x}),x_t(\tilde{x}))\in X$ for all $t\in[0,t^+_{\tilde{x}})$.
Therefore $(t,\tilde{x}) \to (x(t;\tilde{x}),x_t(\tilde{x}))$ defines a map $\Psi:D \to X$ where $D = \{(t,\tilde{x})\,|\,\tilde{x}\in X, t\in[0,t^+_{\tilde{x}})\}$.

Proposition 2.1. Ψ is a local semidynamical system, i.e.

(i) $\Psi(0,\tilde{x}) = \tilde{x}$ for all $\tilde{x}\in X$,

(ii) $\Psi(t_1+t_2;\tilde{x}) = \Psi(t_1;\Psi(t_2;\tilde{x}))$ for all $\tilde{x}\in X$ and all nonnegative t_1,t_2 such that $t_1+t_2 < t^+_{\tilde{x}}$,

(iii) Ψ is continuous on D,

(iv) for any sequence $(\tilde{x}_n)$ such that $\tilde{x}_n \to \tilde{x}\in X$ we have
$$\liminf_{n\to\infty} t^+_{\tilde{x}_n} \geq t^+_{\tilde{x}}$$

Proof. (i) is evident by definition of Ψ and (ii) follows immediately, since equation (2.1) is autonomous. (iv) is a consequence of Proposition 1.2, part b). In order to prove (iii) let $(t_n,\tilde{x}_n)$, n=1,2,... , be a sequence in D with $(t_n,\tilde{x}_n) \to (t,\tilde{x})\in D$. Since $t_n \to t\in[0,t^+_{\tilde{x}})$ it is clear that there exists a number n_0 such that $(t_n,\tilde{x})\in D$ for all $n \geq n_0$.
Proposition 1.2, part c) implies $\|\Psi(t_n;\tilde{x}_n)-\Psi(t_n;\tilde{x})\|_0 \to 0$ as $n \to \infty$. Therefore it is sufficient to show that $\lim_{n\to\infty} \Psi(t_n;\tilde{x}) = \Psi(t;\tilde{x})$. But this is clear by continuity of

$x(t;\tilde{x})$ for $t \geq 0$ and by $\|x_{t_n}(\tilde{x})-x_t(\tilde{x})\|_p \to 0$ as $n \to \infty$.
We shall deal with the operators

$$T_t = \Psi(t;\cdot)\ ,\ t\in[0,t^+),$$

where $t^+ = \sup\limits_{\tilde{x}\in X} t^+_{\tilde{x}}$. If D_t denotes the domain of T_t then it is clear that

$$D_t = \{\tilde{x}\in X \mid (t,\tilde{x})\in D\},\quad t\in[0,t^+).$$

We list the following properties of the family $\{T_t \mid t\in[0,t^+)\}$:

$D_{t_2} \subset D_{t_1}$ for $0 \leq t_1 < t_2 < t^+$, $D_o = X$ and $\tilde{x}\in D_t$ for all $t\in[0,t^+_{\tilde{x}})$. (2.2)

For all $t \geq 0$ the domain D_t is an open subset of X and for each $\delta > 0$ there exists a $t_o > 0$ such that $\{\tilde{x} \mid \|\tilde{x}\|_o \leq \delta\} \subset D_t$ for all $t\in[0,t_o]$. (2.3)

$T_{t_1+t_2}\tilde{x} = T_{t_1}(T_{t_2}\tilde{x})$ for all nonnegative t_1,t_2 such that $t_1+t_2 < t^+$ and all $\tilde{x}\in D_{t_1+t_2}$. (2.4)

For $t\in[0,t^+)$ the operator T_t is continuous on D_t. (2.5)

The map $t \to T_t$ is strongly continuous on $[0,t^+)$, i.e. $t \to T_t\tilde{x}$ is continuous for all $\tilde{x}\in X$ on $[0,t^+_{\tilde{y}})$ and $T_o = I$ (= identity map on X) (2.6)

Properties (2.2), (2.4), (2.5) and (2.6) are trivial consequences of properties proved for Ψ. Property (2.3) is clear from Proposition 1.2, part b) and from Proposition 1.1.

Remark 2.1. Regarding continuity of T_t on D_t we already have proved more than just continuity. Combining Proposition 1.1 and Proposition 1.2, part a), we see that for each δ there exists a $t_o > 0$ such that a Lipschitz condition holds for T_t on $\{\tilde{x} \mid \|\tilde{x}\|_o \leq \delta\}$ provided that $t\in[0,t_o]$. Moreover, a combination of parts a) and b) of Proposition 1.2 shows that

for each $\tilde{x}\in X$ and each $T\in[0,t^+_{\tilde{x}})$ there exists a neighborhood U of $\tilde{x}$ such that a Lipschitz condition holds for T_t on U provided that $t\in[0,T]$. From this by standard compactness arguments we obtain that T_t is Lipschitz on compact subsets of D_t.

<u>Remark 2.2</u>. If (H2) is a global condition then the operators T_t are defined on all of X for all $t \geq 0$ (c.f. Remark 1.7) and $\{T_t | t \geq 0\}$ is a strongly continuous semigroup of continuous operators.

We call $\{T_t | t\in[0,t^+)\}$ the local semigroup associated with equation (2.1). The infinitesimal generator A of this semigroup is defined by

$$A\tilde{x} = \lim_{t\to 0+} \frac{1}{t}(T_t\tilde{x} - \tilde{x}) \ , \ \tilde{x}\in\mathcal{D}(A), \tag{2.7}$$

where $\mathcal{D}(A)$ is the set of all $\tilde{x}\in X$ such that the limit in (2.7) exists.

We shall need the following additional condition on f:

(H3) The map $\tau \to f(x(\tau),x_\tau)$ is continuous on $[0,\alpha)$ for all functions $x(\cdot)$ which are absolutely continuous on $[-r,\alpha)$, where α is any positive number.

<u>Proposition 2.2</u>. Suppose that (H1) - (H3) hold.
Then

a) $\tilde{x} = (\eta,\varphi)\in X$ is in $\mathcal{D}(A)$ if and only if
   (i) $\varphi\in W^{1,p}(-r,0;\mathbb{R}^n)$, i.e. φ is absolutely continuous on $[-r,0]$ and $\dot{\varphi}\in L^p(-r,0;\mathbb{R}^n)$,
   (ii) $\varphi(0) = \eta$.

b) $A\tilde{x} = (f(\varphi(0),\varphi),\dot{\varphi})$ for all $\tilde{x} = (\varphi(0),\varphi)\in\mathcal{D}(A)$.

<u>Proof</u>. Suppose that $\tilde{x} = (\eta,\varphi)$ is in $\mathcal{D}(A)$ and put $A(\eta,\varphi) = (\vartheta,\psi)$. This is equivalent to

$$\lim_{t\to 0+} \frac{1}{t}(x(t;\tilde{x})-\eta) = \vartheta \quad \text{in } \mathbb{R}^n \tag{2.8}$$

and

$$\lim_{t\to 0+} \frac{1}{t}(x_t(\tilde{x})-\varphi) = \psi \text{ in } L^p(-r,0;\mathbb{R}^n). \tag{2.9}$$

Relation (2.9) is equivalent to

$$\lim_{t\to 0+} \int_{-r}^{0} \left|\frac{1}{t}(x_t(\tilde{x})(s)-\varphi(s))-\psi(s)\right|^p ds = 0$$

or to

$$\lim_{t\to 0+} \int_{-r}^{-t} \left|\frac{1}{t}(\varphi(t+s)-\varphi(s))-\psi(s)\right|^p ds = 0 \tag{2.10}$$

and

$$\lim_{t\to 0+} \int_{-t}^{0} \left|\frac{1}{t}(\eta+\int_{0}^{t+s} f(x(\tau;\tilde{x}),x_\tau(\tilde{x}))d\tau - \varphi(s))-\psi(s)\right|^p ds = 0. \tag{2.11}$$

By a characterization of functions in $W^{1,p}(-r,0;\mathbb{R}^n)$ (cf. [5], p.21, or [7], p. 154) relation (2.10) is equivalent to statement (i) in the proposition with $\dot{\varphi} = \psi$.

Relation (2.11) means that for each $\varepsilon > 0$ there exists a $t_o > 0$ such that

$$\left(\int_{-t}^{0}\left|\eta+\int_{0}^{t+s} f(x(\tau;\tilde{x}),x_\tau(\tilde{x}))d\tau-\varphi(s)-t\psi(s)\right|^p ds\right)^{1/p} < \varepsilon t \tag{2.12}$$

for all $t\in[0,t_o]$.

Suppose that φ is continuous and put $\delta = |\varphi(0) - \eta|$. Then for t sufficiently small we have the estimates

$$\begin{aligned}\left(\int_{-t}^{0}\left|\eta+\int_{0}^{t+s} f(x(\tau;\tilde{x}),x_\tau(\tilde{x}))d\tau-\varphi(s)-t\psi(s)\right|^p ds\right)^{1/p} \\ \leq 2\delta t^{1/p} + t\left(\int_{-t}^{0}|\psi(s)|^p ds\right)^{1/p}\end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\left(\int_{-t}^{0}\left|\eta+\int_{0}^{t+s} f(x(\tau;\tilde{x}),x_\tau(\tilde{x}))d\tau - \varphi(s) - t\psi(s)\right|^p ds\right)^{1/p} \\ \geq \frac{1}{2}\delta t^{1/p} - t\left(\int_{-t}^{0}|\psi(s)|^p ds\right)^{1/p} .\end{aligned} \tag{2.14}$$

If $\delta = 0$ then (2.13) shows that (2.12) is true. But if

$\delta > 0$ then by (2.14) we see that (2.12) cannot hold. So (2.13) is equivalent to $\varphi(0) = \eta$ provided that φ is continuous.

Since $x(t;\tilde{x}) = \eta + \int_0^t f(x(\tau;\tilde{x}),x_\tau(\tilde{x}))d\tau$ for $t \geq 0$, relation (2.8) is equivalent to

$$\mathcal{I} = f(\varphi(0),\varphi)$$

provided that φ is absolutely continuous and $\varphi(0) = \eta$. Thus, the proof of Proposition 2.2 is finished.

Remark 2.2. Each paper dealing with semigroup theory and functional-differential equations in the state space X contains a result or a definition corresponding to Proposition 2.2 (cf. for instance [16], [20]). The proof is quite standard and only included in order to make this paper more selfcontained.

Remark 2.3. The additional condition (H3) is not very stringend. In fact all important types of retarded functional-differential equations are incorporated. If we only are interested to prove the result in Proposition 2.2 it is sufficient that $\tau \to f(x(\tau),x_\tau)$ is right-hand continuous at $\tau = 0$ for the class of functions appearing in this additional condition. But the stronger form of this additional condition is needed in order to prove the next result.

Proposition 2.3. Suppose that (H1) - (H3) hold. Then

a) $T_t\tilde{x} \in \mathcal{D}(A)$ for all $\tilde{x} \in \mathcal{D}(A)$ and all $t \in [0,t^+_{\tilde{x}})$.

b) The map $t \to AT_t\tilde{x}$ is continuous on $[0,t^+_{\tilde{x}})$ for all $\tilde{x} \in \mathcal{D}(A)$.

c) $\frac{d}{dt}T_t\tilde{x} = AT_t\tilde{x}$ a.e. on $(0,t^+_{\tilde{x}})$ for all $\tilde{x} \in \mathcal{D}(A)$.

Proof. In order to prove part a) we only have to show that $x_t(\tilde{x}) \in L^p(-r,0;\mathbb{R}^n)$ for all $t \in [0,t^+_{\tilde{x}})$. $\tilde{x} = (\varphi(0),\varphi) \in \mathcal{D}(A)$ implies $\dot{\varphi} \in L^p(-r,0;\mathbb{R}^n)$ and $\tau \to f(x(\tau;\tilde{x}),\dot{x}_\tau(\tilde{x}))$ is in $L^p(0,T;\mathbb{R}^n)$ for all

$T\in(0,t^{+}_{\tilde{x}})$. The proof of part a) is finished if we note that

$$\dot{x}_t(\tilde{x})(s) = \begin{cases} \dot{\varphi}(t+s) \text{ for } t+s < 0, \\ f(x(t+s;\tilde{x}),x_{t+s}(\tilde{x})) \text{ for } t+s \geq 0. \end{cases}$$

In order to prove part b) we write

$$AT_t\tilde{x} = (f(x(t;\tilde{x}),x_t(\tilde{x})),\dot{x}_t(\tilde{x}))$$

By (H3) it is clear that the first component is continuous with respect to t. Continuity of $\dot{x}_t(\tilde{x})$ as a map $[0,t^{+}_{\tilde{x}}) \to L^p(-r,0;\mathbb{R}^n)$ follows from $\dot{x}(\cdot,\tilde{x})\in L^p(-r,T;\mathbb{R}^n)$ for all $T\in[0,t^{+}_{\tilde{x}})$ and the continuity the L^p-norm with respect to translation.

In order to proove part c) we first have to show that the map $t \to T_t\tilde{x}$ is differentiable a.e. on $(0,t^{+}_{\tilde{x}})$ provided that $\tilde{x}\in\mathcal{D}(A)$. This is trivial for the first component $x(t;\tilde{x})$ of $T_t\tilde{x}$. With respect to the second component $x_t(\tilde{x})$ one has to show that

$$\lim_{h\to 0} \|\tfrac{1}{h}(x_{t+h}(\tilde{x})-x_t(\tilde{x})) - \dot{x}_t(\tilde{x})\|_p = 0.$$

We leave this to the reader. Again, one has to use the result given in [7], p. 154.

<u>Remark 2.4</u>. Proposition 2.3 shows that with equation (2.1) we can associate an abstract evolution equation in X. Moreover, $T_t\tilde{x}$ is a strong solution of this evolution equation for $\tilde{x}\in\mathcal{D}(A)$.

$\mathcal{D}(A)$ is a linear subspace of X not dependent on f. It can be characterized as that set of initial data $\tilde{x}$ corresponding to solutions $x(t;\tilde{x})$ which are absolutely continuous with L^p-integrable derivative on compact subintervals of $[-r,t^{+}_{\tilde{x}})$. If we take the state space $C(-r,0;\mathbb{R}^n)$ instead of X then $\mathcal{D}(A)$ is dependent on f and is given by $\{\varphi\in C(-r,0;\mathbb{R}^n) \mid \dot{\varphi}\in C(-r,0;\mathbb{R}^n)$ and $\dot{\varphi}(0) = f(\varphi(0),\varphi)\}$ whereas $A\varphi = \dot{\varphi}$. Here, $\mathcal{D}(A)$ can be

characterized as the set of those initial data $\tilde{x}$ corresponding to solutions $x(t;\tilde{x})$ which have a continuous derivative on $[-r,t^{+}_{\tilde{x}})$ (cf. for instance [19]).

Proposition 2.2 shows that the operator A can be written as

$$A = A_0 + A_1 \qquad (2.15)$$

where

$$A_0(\varphi(0),\varphi) = (0,\dot{\varphi})$$

and

$$A_1(\varphi(0),\varphi) = (f(\varphi(0),\varphi),0)$$

for all $(\varphi(0),\varphi)\in\mathcal{D}(A)$. Of course, A_1 is defined on all of

$$\mathcal{C} = \{(\varphi(0),\varphi)\,|\,\varphi\in C(-r,0;\mathbb{R}^n)\}.$$

$\mathcal{C}$ will be considered as a Banach-space with norm $\|(\varphi(0),\varphi)\|_{\mathcal{C}} = \|\varphi\|_C$.

A_0 is the infinitesimal generator of the linear semigroup $\{S_t | t \geq 0\}$ generated by the solutions of $\dot{x} = 0$, i.e. $S_t(\eta,\varphi) = (\eta,\psi)$ where

$$\psi(s) = \begin{cases} \eta & \text{for } t+s \geq 0, \\ \varphi(t+s) & \text{for } t+s < 0. \end{cases}$$

Since $\mathcal{D} = \mathcal{D}(A) = \mathcal{D}(A_0)$ we have from linar semigroup theory (see for instance [21])

<u>Proposition 2.4.</u> $\mathcal{D}$ is dense in X.

Splitting up the infinitesimal generator A as in (2.15) shows that properties of f in a straightforward way lead to properties of A_1 and therefore of A. This is one advantage of taking X as the state space. As we have stated above $\mathcal{D}(A)$ is dependent on the right-hand side of equation (2.1) and properties of f are reflected in $\mathcal{D}(A)$ if we take the state space $C(-r,0;\mathbb{R}^n)$.

& 4. Representation of the solution semigroup, global case

Throughout this section we assume that (H1) - (H3) hold for f and that (H2) is a global condition. We shall make use of some results of nonlinear semigroup theory (as a reference see for instance [5]).

If Y is a Banach space and Y^* is its dual then the duality map F on Y is defined by

$$F(x) = \{x^* \in Y^* \mid \|x\|^2 = \|x^*\|^2 = \langle x, x^* \rangle\}$$

for $x \in Y$, where $\langle \cdot , \cdot \rangle$ denotes the pairing between elements of Y and Y^*. If Y^* is strictly convex then F is univalent. An operator $A \subset Y \times Y$ is called dissipative if $\langle u_1 - u_2, x^* \rangle \leq 0$ for all pairs $[x_i, u_i] \in A$, i=1,2, and some $x^* \in F(x_1 - x_2)$. If A and F are univalent, the dissipativeness condition is

$$\langle Ax_1 - Ax_2 \, , \, F(x_1 - x_2) \rangle \leq 0 \tag{3.1}$$

for all $x_i \in \mathcal{D}(A)$, i=1,2.

An equivalent characterization of an dissipative operator was given by Kato [14]: $A - \omega I$ is dissipative if and only if

$$\|x_1 - \lambda u_1 - (x_2 - \lambda u_2)\| \geq (1 - \omega\lambda)\|x_1 - x_2\| \tag{3.2}$$

for all $[x_i, u_i] \in A$ and all $\lambda \in (0, \frac{1}{\omega})$. If $(I - \lambda A)^{-1}$ exists on Y, then (3.2) implies

$$\|(I - \lambda A)^{-1} x_1 - (I - \lambda A)^{-1} x_2\| \leq \frac{1}{1 - \omega\lambda} \|x_1 - x_2\|$$

for $x_i \in Y$, i=1,2, and $\lambda \in (0, \frac{1}{\omega})$.

The spaces X supplied with the norm $\|(\vartheta, \psi)\| = (|\vartheta|^p + \|\psi\|_p^p)^{1/p}$ and $X^* = \mathbb{R}^n \times L^q(-r, 0; \mathbb{R}^n)$, $\frac{1}{p} + \frac{1}{q} = 1$, supplied with the dual norm are uniformly convex. Therefore the duality map on X is univalent.

In [20], Proposition 2.1, it was proved that the duality map $F(\tilde{x})$ for $\tilde{x} = (\eta,\varphi)\in X$ is given by

$$\langle(\vartheta,\psi),F(\tilde{x})\rangle \tag{3.3}$$
$$= \|\tilde{x}\|^{2-p}\left(\vartheta^T\eta|\eta|^{p-2}+\int_{-r}^{0}\psi(s)^T\varphi(s)|\varphi(s)|^{p-2}ds\right)$$

for all $(\vartheta,\psi)\in X$.

With respect to the linear operator A_0 we have the following results.

<u>Proposition 3.1.</u>

a) $(I-\lambda A_0)^{-1}$ exists on X for all $\lambda \geq 0$ and is given by

$$(I-\lambda A_0)^{-1}(\vartheta,\psi) = \left(\vartheta, e^{\cdot/\lambda}\left(\vartheta-\frac{1}{\lambda}\int_0^{\cdot} e^{-\tau/\lambda}\psi(\tau)d\tau\right)\right) \tag{3.4}$$

for all $(\vartheta,\psi)\in X$ and $\lambda > 0$.

b) $A_0-\frac{1}{p}I$ is dissipative and therefore

$$\|(I-\lambda A_0)^{-1}\tilde{x}\| \leq \frac{1}{1-\lambda/p}\|\tilde{x}\|$$

obtains for all $\tilde{x}\in X$ and $\lambda\in(0,p)$.

c) The restriction of $(I-\lambda A_0)^{-1}$ to the Banach space $\mathcal{C}$ is a contraction, i.e.

$$\|(I-\lambda A_0)^{-1}\tilde{x}\|_{\mathcal{C}} \leq \|\tilde{x}\|_{\mathcal{C}}$$

for all $\tilde{x} \in \mathcal{C}$.

<u>Proof</u>. The semigroup S_t, $t \geq 0$, with infinitesimal generator A_0 is equicontinuous, i.e. for some constant $M > 0$ we have $\|S_t\| \leq M$ for all $t \geq 0$. Therefore existence of $(I-\lambda A_0)^{-1}$ for all $\lambda > 0$ immediately follows from linear semigroup theory. The representation (3.4) is obtained by solving the equation $(I-\lambda A_0)(\eta,\varphi) = (\vartheta,\psi)$, $(\vartheta,\psi)\in X$, which is equivalent to $\eta = \vartheta$ and $\varphi(s) - \lambda\dot{\varphi}(s) = \psi(s)$, $s\in[-r,0]$.

With respect to part b) we refer to Proposition 3.1 of [20]. Part c) immediately follows from the fact that the restriction of the semigroup S_t , $t \geq 0$, to the space $\mathcal{C}$ is a contraction semigroup. But the result can also easily be obtained by direct computation using (3.4).

Remark 3.1. From Proposition 3.1, part a), we infer that A_0 is m-dissipative, i.e $\mathcal{R}(I-\lambda A_0) = X$ for all $\lambda > 0$.

As stated above we deal with the case where (H2) is a global condition, i.e. the function $\gamma(t)$ is not dependent on β. For $\varepsilon > 0$ we define the operator A_ε with $\mathcal{D}(A_\varepsilon) = \mathcal{D}$ by

$$A_\varepsilon(\varphi(0),\varphi) = \left(\frac{1}{\varepsilon}\int_0^\varepsilon f(S_s(\varphi(0),\varphi))ds \ , \ \dot{\varphi}\right) \tag{3.5}$$

A_ε corresponds to equation

$$\dot{x}(t) = \frac{1}{\varepsilon}\int_0^\varepsilon f(S_s(x(t),x_t))ds = f_\varepsilon(x(t),x_t) \tag{3.6}$$

in the same way as A corresponds to equation (1.1).

Note, that f_ε is defined on all of X, if for $(\eta,\varphi)\in X$ we put $f_\varepsilon(\eta,\varphi) = \frac{1}{\varepsilon}\int_0^\varepsilon f(S_s(\eta,\varphi))ds$. If (H3) holds for f then it is also true for f_ε.

Proposition 3.2. Suppose, that the function $\gamma(t)$ in (H2) does not depend on β.

a) The operator $A_\varepsilon-\omega I$ is dissipative with $\omega = \frac{1}{p}+\frac{1}{\varepsilon}\gamma(\varepsilon)$, $0 < \varepsilon \leq 1$, and $\omega = \frac{1}{p} + \varepsilon^{1/p-1}\gamma(\varepsilon)$, $\varepsilon > 1$.

b) The solutions $x^{(\varepsilon)}(t;\tilde{x})$ of equation (3.6) exist globally for all $\tilde{x}\in X$ and $(x^{(\varepsilon)}(t;\tilde{x}),x_t^{(\varepsilon)}(\tilde{x})) = T_t^{(\varepsilon)}\tilde{x}$ where

$$T_t^{(\varepsilon)}\tilde{x} = \lim_{n\to\infty} \left(I-\frac{t}{n}A_\varepsilon\right)^{-n}\tilde{x} \ , \tag{3.7}$$

the limit existing uniformly on compact intervals.

c) $T_t^{(\varepsilon)}\tilde{x}$ is the unique solution of

$$T_t^{(\varepsilon)}\tilde{x} = S_t\tilde{x} + \int_0^t S_{t-s}A_{\varepsilon,1}T_s^{(\varepsilon)}\tilde{x}\,ds \ , \ t \geq 0 \ , \tag{3.8}$$

where $\tilde{x}\in X$ and $A_{\varepsilon,1} = A_\varepsilon - A_o$.

<u>Proof</u>. Choose $\tilde{x}_i = (\varphi_i(0),\varphi_i)\in\mathcal{D}$, i=1,2. Then by (3.3) we get

$$\langle A_{1,\varepsilon}\,\tilde{x}_1 - A_{1,\varepsilon}\tilde{x}_2 \ , \ F(\tilde{x}_1-\tilde{x}_2)\rangle$$

$$= \frac{1}{\varepsilon}\|\tilde{x}_1-\tilde{x}_2\|^{2-p}|\varphi_1(0)-\varphi_2(0)|^{p-2}(\varphi_1(0)-\varphi_2(0))^T\int_0^\varepsilon(f(S_s(\varphi_1(0),\varphi_1) - f(S_s(\varphi_2(0),\varphi_2))ds$$

$$\leq \frac{1}{\varepsilon}\,\gamma(\varepsilon)\|\tilde{x}_1-\tilde{x}_2\|(\varepsilon|\varphi_1(0)-\varphi_2(0)|^p + \int_{-r}^0|\varphi_1(s)-\varphi_2(s)|^p ds)^{1/p}$$

$$\leq \begin{cases} \frac{1}{\varepsilon}\gamma(\varepsilon)\|\tilde{x}_1-\tilde{x}_2\|^2 & \text{for } 0 < \varepsilon \leq 1, \\ \varepsilon^{1/p\,-1}\gamma(\varepsilon)\|\tilde{x}_1-\tilde{x}_2\|^2 & \text{for } \varepsilon > 1. \end{cases}$$

This estimate and the dissipativeness of $A_o-\frac{1}{p}I$ implies the desired result.

Since $A_\varepsilon-\omega I$ isdissipative, an application of Theorem I in [9] shows that A_ε is the infinitesimal generator of the semigroup $T_t^{(\varepsilon)}$ given by (3.7) provided that $\mathcal{R}((I-\ A_\varepsilon)^{-1}) = X = \overline{\mathcal{D}}$ for all sufficiently small λ. In order to verify this we have to discuss the equation $\tilde{x} = (I-\lambda A_\varepsilon)\tilde{y}$, $\tilde{y}\in\mathcal{D}$, $\tilde{x}\in X$, which is equivalent to the fixed point equation

$$\tilde{y} = (I-\lambda A_o)^{-1}\tilde{x} + \lambda(I-\lambda A_o)^{-1}A_\varepsilon\tilde{y}$$

for $\tilde{y}$. It is easy to see that this fixed point equation has a unique solution $\tilde{y}\in\mathcal{D}$ for all $\tilde{x}\in X$ provided that $\lambda\in(0,\frac{1}{\omega})$. It was shown in [20], Proposition 5.9, that $(x^{(\varepsilon)}(t;\tilde{x})\ ,\ x_t^{(\varepsilon)}(\tilde{x})) = T_t^{(\varepsilon)}x$ for all $\tilde{x}\in X$ and $t \geq 0$.

Formula (3.7) also follows from the results given by Webb in [18]. There it is also proved that $T_t^{(\varepsilon)}\tilde{x}$ is the unique solution of (3.8).

Remark 3.2. It is immediate that

$$\lim_{\varepsilon\to 0+} A_\varepsilon \tilde{x} = A\tilde{x} \text{ for all } \tilde{x} \in \mathcal{D}.$$

If $\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\gamma(\varepsilon) = \gamma'(0)$ exists then from the proof of Proposition 3.2 we obtain

$$\langle A\tilde{x}_1 - A\tilde{x}_2 , F(\tilde{x}_1-\tilde{x}_2)\rangle \leq (\frac{1}{p}+ \gamma'(0))\|\tilde{x}_1-\tilde{x}_2\|^2$$

for all $\tilde{x}_i \in \mathcal{D}$, $i=1,2$, i.e. $A-\omega I$ is dissipative with $\omega=\frac{1}{p}+\gamma'(0)$. But in general $\gamma'(0)$ does not exist. The only case where $\gamma'(0)$ exists seems to obtain when condition (1.5) holds for f (with L independent of M). In this case $\gamma'(0) = L$. But then an easy calculation directly shows that $A-(\frac{1}{p}+L)I$ is dissipative. In case of the difference-differential equation (1.3) $\gamma'(0)$ does not exist, for instance.

Remark 3.3. If the right-hand side of equation (2.1) is of special form, then it was shown in [20] that $A-\omega I$ is dissipative for some $\omega \geq 0$ provided one uses weighted norms. The weighting function then is defined by the right-hand side of equation (2.1).

Remark 3.4. In order to prove existence of solutions for equation (3.6) it is not necessary to use the Crandall-Liggett-Theorem and Proposition 5.9 of [20]. From the properties of f it follows that f_ε is defined on all of X and is globally Lipschitz with Lipschitz constant $\frac{1}{\varepsilon}\gamma(\varepsilon)$ for $0 < \varepsilon \leq 1$. By direct verification one can show that also conditions (H1) and (H2) hold for equation (3.6). A corresponding function $\gamma(t)$ in (H2) is $\gamma^{(\varepsilon)}(t) = \frac{1}{\varepsilon}\gamma(\varepsilon)t(1+(\frac{\varepsilon}{t})^{1/p})$, $0 < \varepsilon \leq 1$.
Thus, existence and uniqueness of solutions to equation (3.6) follow from the results of Section 1.

But the functions $\gamma^{(\varepsilon)}(t)$, are dependent on ε and cannot be estimated by an function independent on ε as $\varepsilon \to 0$.

In order to prove that the semigroups $T_t^{(\varepsilon)}$ approximate the solution semigroup T_t we need additional conditions:

(H4) f is locally Lipschitz on $\mathcal{C}$, i.e. for all $\beta > 0$ there exists a constant $L(\beta) > 0$ such that

$$|f(\varphi(0),\varphi) - f(\psi(0),\psi)| \leq L(\beta) \cdot \|\varphi-\psi\|_C$$

provided that $\|\varphi\|_C \leq \beta$ and $\|\psi\|_C \leq \beta$.

(H5) For all $\beta > 0$ there exists a function $\gamma_\beta(t)$ which can be used in condition (H2) not only for equation (2.1) but also for equation (3.6) if ε is sufficiently small.

<u>Remark 3.5</u>. Condition (H4) obviously implies (H3). In case of equation (1.3) conditions (H4) and (H5) are true without any further assumption on h. This is clear with respect to (H4). In order to prove that (H5) is true one has first to interchange the order of integration and then to proceed in the same way as in Remark 1.2. We see that the function $\gamma_\beta(t)$ given there can be used in (H5) for $\varepsilon \in (0,k_1]$.

If (1.5) holds for f then again (H4) and (H5) are true without further restrictions for f.

<u>Remark 3.6</u>. If condition (H4) is true for f then it is also true for the equations (3.6) with the same Lipschitz constant $L(\beta)$. This follows from the estimate

$$|f_\varepsilon(\varphi(0),\varphi)-f_\varepsilon(\psi(0),\psi)| \leq \frac{1}{\varepsilon}\int_0^\varepsilon |f(S_s(\varphi(0),\varphi)-f(S_s(\psi(0),\psi))|ds$$

$$\leq \frac{1}{\varepsilon}L(\beta)\int_0^\varepsilon \|S_s(\varphi(0)-\psi(0),\varphi-\psi)\|_{\mathcal{C}}\, ds$$

$\leq L(\beta)\|\varphi-\psi\|_C, \beta = \max(\|\varphi\|_C, \|\psi\|_C)$, where we have used the fact that S_t restricted to $\mathcal{C}$ is a contraction.

<u>Proposition 3.3</u>. Suppose that conditions (H1) - (H5) hold for f and that the function $\gamma(t)$ appearing in (H5) can be chosen

independent of β. Then

$$\lim_{\varepsilon\to 0+} T_t^{(\varepsilon)}\tilde{x} = T_t\tilde{x}$$

for all $\tilde{x}\in X$, the limit being uniform on compact intervals.

Proof. It is sufficient to prove $\lim_{\varepsilon\to 0+} x^{(\varepsilon)}(t)=x(t)$ uniformly on compact intervals, where $x(t)=x(t;\tilde{x})$ and $x^{(\varepsilon)}(t)=x^{(\varepsilon)}(t;\tilde{x})$ are the solutions of equation (2.1) and (3.6) with initial data $\tilde{x}$, respectively.

We first consider the case $\tilde{x}\in\mathcal{C}$. By assumption (H4) we have

$$\frac{1}{\varepsilon}\int_0^t f(S_s(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)}))ds = f(S_{s*}(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)})) \qquad (3.9)$$

for $\tau \geq 0$ where $s^* = s^*(\tau)\in(0,\varepsilon)$.

We see from Remark 3.6 that the set $\{x^{(\varepsilon)}(\cdot) \mid \varepsilon > 0\}$ is an equicontinuous set of functions on each interval $[-r,T]$, $T > 0$. Therefore

$$\int_0^t \|(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)}) - S_{s*}(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)})\|_{\mathcal{C}}\, d\tau \leq h(\varepsilon;t) \qquad (3.10)$$

for $\varepsilon > 0$, $t \geq 0$, where $\lim_{\varepsilon\to 0+} h(\varepsilon;t) = 0$.

From (3.9) and (3.10) we get the estimate

$$|x(t) - x^{(\varepsilon)}(t)| \leq \int_0^t |f(x(\tau),x_\tau) - f(S_{s*}(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)}))|d\tau$$

$$\leq \int_0^t |f(x(\tau),x_\tau) - f(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)})|d\tau$$

$$+ \int_0^t |f(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)}) - f(S_{s*}(x^{(\varepsilon)}(\tau),x_\tau^{(\varepsilon)}))|d\tau$$

$$\leq L(\delta)\int_0^t \|x_\tau - x_\tau^{(\varepsilon)}\|_C d\tau + L(\delta)h(\varepsilon;T)$$

for $t\in[0,T)$, $T > 0$, where δ is a uniform bound for $x(\cdot)$ and $x^{(\varepsilon)}(\cdot)$, $\varepsilon > 0$, on $[-r,T+\varepsilon]$. An application of Gronwall's

inequality immediately gives

$$\|x_t - x_t^{(\varepsilon)}\|_C \leq L(\delta)h(\varepsilon;T)e^{tL(\delta)}$$

for $t\in[0,T]$. This shows

$$\lim_{\varepsilon\to 0+} \|T_t^{(\varepsilon)}\tilde{x} - T_t\tilde{x}\| = 0 , \quad \tilde{x}\in \mathcal{C} , \tag{3.11}$$

uniformly on compact intervals.

Now, choose $\tilde{x}\in X$ and a sequence $\{\tilde{x}_k\}$ in $\mathcal{C}$ such that $\tilde{x}_k\to \tilde{x}$ in X.

Take a sequence $\varepsilon_n \to 0$.

Assumption (H5) and Proposition 1.2, part a), together with Remark 1.8 imply that for any $\mu \geq 0$ we can choose a number k_0 such that

$$|x^{(\varepsilon_n)}(t) - x^{(\varepsilon_n)}(t;\tilde{x}_k)| \leq \mu,$$

$$|x(t;\tilde{x}_k) - x(t)| \leq \mu$$

for all n and all $t\in[0,T]$ provided that $k \geq k_0$. Therefore we obtain

$$|x^{(\varepsilon_n)}(t) - x(t)| \leq 2\mu + |x^{(\varepsilon_n)}(t;\tilde{x}_k) - x(t;\tilde{x}_k)| \quad \text{for}$$

all $n, t\in[0,T]$ and $k \geq k_0$.

By (3.11) for each k we can choose n_k such that

$$|x^{(\varepsilon_{n_k})}(t;\tilde{x}_k) - x(t;\tilde{x}_k)| \leq \frac{1}{k}$$

for $t\in[0,T]$. Therefore

$$\lim_{k\to\infty} x^{(\varepsilon_{n_k})}(t;\tilde{x}) = x(t)$$

uniformly on compact intervals. Since $\varepsilon_n \to 0$ was arbitrary this proves the desired result.

Corollary. Under the conditions of Proposition 3.3 $T_t\tilde{x}, \tilde{x} \in X$, is the unique solution of

$$T_t\tilde{x} = S_t\tilde{x} + \int_0^t S_{t-s} A_1 T_s\tilde{x} ds , \quad t \geq 0. \tag{3.12}$$

Proof. In view of formula (3.8) and the uniform convergence of $T_t^{(\varepsilon)}\tilde{x}$ on compact intervals we only have to prove

$$\lim_{\varepsilon\to 0+} \int_0^t S_{t-s} A_{1,\varepsilon} T_s^{(\varepsilon)}\tilde{x} \, ds = \int_0^t S_{t-s} A_1 T_s \tilde{x} \, ds. \tag{3.13}$$

First we consider $\tilde{x} \in \mathcal{C}$. Then

$$\begin{aligned} \|A_{1,\varepsilon} T_s^{(\varepsilon)}\tilde{x} - A_1 T_s\tilde{x}\| &= |f_\varepsilon(T_s^{(\varepsilon)}\tilde{x}) - f(T_s\tilde{x})| \\ &= |f(S_{\tau(s)} T_s^{(\varepsilon)}\tilde{x}) - f(T_s\tilde{x})| \\ &\leq L(\delta) \|S_{\tau(s)} T_s^{(\varepsilon)}\tilde{x} - T_s\tilde{x}\|_{\mathcal{C}} \\ &\leq L(\delta) \left\{ \|S_{\tau(s)} T_s^{(\varepsilon)}\tilde{x} - S_{\tau(s)} T_s\tilde{x}\|_{\mathcal{C}} + \|S_{\tau(s)} T_s\tilde{x} - T_s\tilde{x}\|_{\mathcal{C}} \right\} \end{aligned}$$

where $\tau(s) \in [0,\varepsilon]$ and $\delta = \sup_{[0,T]} (\|T_s^{(\varepsilon)}\tilde{x}\|_{\mathcal{C}}, \|T_s\tilde{x}\|_{\mathcal{C}})$.

This estimate shows that

$$\lim_{\varepsilon\to 0} \|A_{1,\varepsilon} T_s^{(\varepsilon)}\tilde{x} - A_1 T_s\tilde{x}\| = 0$$

uniformly on $[0,t)$. This proves (3.13) in case $\tilde{x} \in \mathcal{C}$.

Now take $\tilde{x} \in X$ and choose a sequence $(\tilde{x}_k)$ in $\mathcal{C}$ such that $\tilde{x}_k \to \tilde{x}$.

For each k we have

$$T_t\tilde{x}_k = S_t\tilde{x}_k + \int_0^t S_{t-s} A_1 T_s \tilde{x}_k ds.$$

Recalling the definitions of S_t and A_1 we see that for $\tilde{y} \in X$

$$S_{t-s} A_1 T_s \tilde{y} = (f(T_s\tilde{y}), \psi_s(\tilde{y}))$$

where $\psi_s(\tilde{y})$ denotes the function in $L^p(-r,0;\mathbb{R})$ defined by

$$\psi_s(\tilde{y})(\Theta) = \begin{cases} f(T_s\tilde{x}) & \text{for } \Theta \geq s-t, \\ 0 & \text{for } \Theta > s-t. \end{cases}$$

Therefore

$$\int_0^t (S_{t-s}A_1T_s\tilde{x} - S_{t-s}A_1({}_s\tilde{x}_k)ds$$

$$= (\int_0^t (f(T_s\tilde{x}) - f(T_s\tilde{x}_k))ds, \int_0^t (\psi_s(\tilde{x}) - \psi_s(\tilde{x}_k))ds).$$

For the first component we get

$$|\int_0^t [f(T_s\tilde{x}) - f(T_s\tilde{x}_k)]ds| \leq \gamma_\beta(t)(\int_{-r}^t |x(s;\tilde{x}) - x(s;\tilde{x}_k)|^p ds)^{1/p} \quad (3.14)$$

where $\gamma_\beta(t)$ is appropriately chosen according to (H2).

For the second component we have

$$\|\int_0^t [\psi_s(\tilde{x}) - \psi_s(\tilde{x}_k)]ds\|_p$$

$$\leq \int_0^t (\int_{-r}^0 (\psi_s(\tilde{x})(\Theta) - \psi_s(\tilde{x}_k)(\Theta)|^p d\Theta)^{1/p} ds$$

$$\leq \int_0^t (\int_{-r}^0 |f(T_s\tilde{x}) - f(T_s\tilde{x}_k)|^p d\Theta)^{1/p} ds$$

$$\leq r^{1/p} \gamma_\beta(t)(\int_{-r}^t |x(s;\tilde{x}) - x(s;\tilde{x}_k)|^p ds)^{1/p} \quad (3.15)$$

The estimates (3.14) and (3.15) together with Proposition 1.2 prove that

$$\lim_{k\to\infty} \int_0^t S_{t-s}A_1T_s\tilde{x}_k ds = \int_0^t S_{t-s}A_1T_s\tilde{x}ds.$$

This finishes the proof of the corollary.

& 5. Representation of the solution semigroup, local case

In this section we remove the assumption that (H5) is a global condition and show that the results of Section 3 remain valid with obvious modifications.

The operators A_ε, $A_{1,\varepsilon}$ and the map f_ε are defined as in Section 3. We need

Lemma 4.1. For each $\beta > 0$ and $\rho > 0$ there exists a constant $\lambda_0 = \lambda_0(\beta,\rho)$ such that for all $\lambda\in(0,\lambda_0)$ and all $\tilde{x}\in X$ with $\|\tilde{x}\| \le \beta$ the equation

$$(I - \lambda A_\varepsilon)\tilde{y} = \tilde{x} \tag{4.1}$$

has a unique solution $\tilde{y}\in\mathcal{D}$ such that $\|\tilde{y}\| \le \beta+\rho$.

Proof. The equation is equivalent to the fixed point equation

$$\tilde{y} = (I-\lambda A_0)^{-1}\tilde{x} + \lambda(I-\lambda A_0)^{-1}A_{1,\varepsilon}\tilde{y}.$$

Then the result follows by standard arguments using the dissipativeness of A_0 and the Lipschitz continuity of $A_{1,\varepsilon}$.

If for $\tilde{x}\in X$ there exists a ball with center in the origin such that equation (4.1) has a unique solution $\tilde{y}$ in this ball then we put $\tilde{y} = (I - \lambda A_\varepsilon)^{-1}\tilde{x}$. The lemma shows that $(I - \lambda A_\varepsilon)^{-1}\tilde{x}$ exists for λ sufficiently small. In general equation (4.1) does not have a unique solution in all of $\mathcal{D}$.

Proposition 4.1. Suppose that (H1) - (H3) are true and define $T_t^{(\varepsilon)}\tilde{x} = (x^{(\varepsilon)}(t;\tilde{x}),x_t^{(\varepsilon)}(\tilde{x}))$ where $x^{(\varepsilon)}(t;\tilde{x})$ is the solution of (3.6) with initial data $\tilde{x}$. Then

$$T_t^{(\varepsilon)}\tilde{x} = \lim_{n\to\infty} (I - \tfrac{t}{n}A_\varepsilon)^{-n}\tilde{x} \tag{4.2}$$

for all $\tilde{x}\in X$ and $t\in[0,t_{\tilde{x}}^{+})$ the convergence being uniform on

compact subintervals of $[0,t^+_{\tilde{x}})$. Moreover, $T_t^{(\varepsilon)}\tilde{x}$ is the unique solution of

$$T_t^{(\varepsilon)}\tilde{x} = S_t\tilde{x} + \int_0^t S_{t-s}A_{1,\varepsilon}T_s^{(\varepsilon)}\tilde{x}ds \tag{4.3}$$

on $[0,t^+_{\tilde{x}})$.

Proof. The existence and uniqueness of solutions to equation (3.6) is clear by the results of Section 1. We choose $T\in(0,t^+_{\tilde{x}})$ and $\beta > 0$ such that $\|T_t^{(\varepsilon)}x\| \leq \beta-1$ on $[0,T]$.

Next we define

$$\hat{f}_\varepsilon(\tilde{x}) = f_\varepsilon(\pi\,\tilde{x})\ ,\ \tilde{x}\in X,$$

where π is the radial projection

$$\pi\tilde{x} = \begin{cases} \tilde{x} & \text{for } \|\tilde{x}\| \leq \beta, \\ \frac{\beta}{\|\tilde{x}\|}\tilde{x} & \text{for } \|\tilde{x}\| > \beta. \end{cases}$$

π is globally Lipschitzean with Lipschitz constant 2 (cf. [10]). Therefore $\hat{f}_\varepsilon$ is globally Lipschitzean with constant $\frac{2}{\varepsilon}\gamma_\beta(\varepsilon)$. If we define $\hat{A}_{1,\varepsilon}\tilde{x} = (\hat{f}_\varepsilon(\tilde{x}),0)$ and $\hat{A}_\varepsilon = A_0 + \hat{A}_{1,\varepsilon}$, then $\hat{A}_\varepsilon - \alpha I$ is dissipative with $\alpha = \frac{1}{p} + \frac{2}{\varepsilon}\gamma_\beta(\varepsilon)$. By Remark 3.4 we see that Proposition 3.2 applies and we get

$$T_t^{(\varepsilon)}\tilde{x} = \lim_{n\to\infty}(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-n}\tilde{x} \tag{4.4}$$

uniformly on $[0,T]$. Note, that the solution $\hat{x}^{(\varepsilon)}(t;\tilde{x})$ of $\dot{x}(t) = \hat{f}_\varepsilon(x(t),x_t)$ coincides with $x^{(\varepsilon)}(t;\tilde{x})$ on $[0,T]$.

In order to finish the proof we have to show that in (4.4) we can replace $(I - \frac{t}{n}\hat{A}_\varepsilon)^{-n}$ by $(I - \frac{t}{n}A_\varepsilon)^{-n}$ provided that n is sufficiently large. To this end we only need to prove that $\|(I - \frac{t}{n}\hat{A}_\varepsilon)^{-j}\tilde{x}\| \leq \beta$, $j=1,\dots,n$, for all $t\in[0,T]$ and $n \geq n_0$, n_0 some positive integer larger than the constant $\lambda_0(\beta,1)$ of Lemma 4.1.

We first suppose that $\tilde{x} \in \mathcal{D}$ and write

$$(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-j}\tilde{x} - T^{(\varepsilon)}_{jt/n}\tilde{x} =$$

$$\sum_{\kappa=0}^{j-1}(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-j+\kappa}(T^{(\varepsilon)}_{\kappa t/n}\tilde{x} - (I - \tfrac{t}{n}\hat{A}_\varepsilon)T^{(\varepsilon)}_{(\kappa+1)t/n}\tilde{x})$$

$$= \sum_{\kappa=0}^{j-1}(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-j+\kappa}\int_0^{t/n}(\hat{A}_\varepsilon T^{(\varepsilon)}_{(\kappa+1)t/n}\tilde{x} - \hat{A}_\varepsilon T_{\kappa t/n+s}\tilde{x})ds.$$

This and the dissipativeness of $\hat{A}_\varepsilon - \alpha I$ give the estimate

$$\|(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-j}\tilde{x} - T^{(\varepsilon)}_{jt/n}\tilde{x}\| \leq e^{\alpha t}\sum_{\kappa=0}^{j-1}\int_0^{t/n}\|\hat{A}_\varepsilon T^{(\varepsilon)}_{(\kappa+1)t/n}\tilde{x} - \hat{A}_\varepsilon T^{(\varepsilon)}_{\kappa t/n}\tilde{x}\|ds.$$

From Proposition 2.3, part b) there exists for each positive number μ a number n_0 such that

$$\|\hat{A}_\varepsilon T^{(\varepsilon)}_{(\kappa+1)t/n}\tilde{x} - \hat{A}_\varepsilon T^{(\varepsilon)}_{\kappa t/n+s}\tilde{x}\| < \mu$$

for $n \geq n_0$, $s \in [0, t/n]$ and $\kappa = 0,\dots,n-1$. Thus we arrive at

$$\|(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-j}\tilde{x} - T^{(\varepsilon)}_{jt/n}\tilde{x}\| \leq \mu e^{\alpha t}\frac{jt}{n} \leq \mu t e^{\alpha t} < 1$$

for $n \geq n_0$, $j = 1,\dots,n$ and $t \in [0,T]$ provided that μ is chosen such that $\mu T e^{\alpha T} < 1$. By choice of β the last estimate proves $\|(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-j}\tilde{x}\| < \beta$ on $[0,T]$ for $n \geq n_0$ and $j = 1,\dots,n$.

If $\tilde{x}$ is not in $\mathcal{D}$ the result follows from the densitiy of $\mathcal{D}$ in X and the Lipschitz continuity of $(I - \tfrac{t}{n}\hat{A}_\varepsilon)^{-1}$.

Proposition 4.2. Suppose that (H1) - (H5) are true. Then for all $\tilde{x} \in X$

$$\lim_{\varepsilon \to 0+} T^{(\varepsilon)}_t\tilde{x} = T_t\tilde{x}$$

uniform on compact subintervals of $[0, t^{+}_{\tilde{x}})$. Moreover, $T_t\tilde{x}$ is the unique solution of

$$T_t\tilde{x} = S_t\tilde{x} + \int_0^t S_{t-s}A_1T_s\tilde{x}\,ds\ ,\ t+[0,t^{+}_{\tilde{x}}).$$

Proof. We first prove that for $T\in(0,t^+_{\tilde{x}})$ there exists a number $\varepsilon_0 > 0$, such that the solution $x^{(\varepsilon)}(t;\tilde{x})$ of equation (3.6) with initial data $\tilde{x}$ exists on $[0,T]$ for $0 < \varepsilon \leq \varepsilon_0$. Moreover, the functions $x^{(\varepsilon)}(t;\tilde{x})$ are uniformly bounded on $[0,T]$ for $0 < \varepsilon \leq \varepsilon_0$. Indeed, choose δ such that $\|T_t\tilde{x}\| < \delta - 3/2$ on $[0,T]$. Assumption (H5) and Proposition 1.1 imply that there exists a constant $\alpha > 0$ such that $x^{(\varepsilon)}(t;\tilde{x})$ exists on $[0,\alpha]$ and $\|T_t^{(\varepsilon)}\tilde{x}\| < \delta - 1/2$ on $[0,\alpha]$ for all $\varepsilon > 0$. Note, that $f_\varepsilon(0,0) = f(0,0)$. Then quite analogous to the proof of Proposition 3.3 we see that $T_t^{(\varepsilon)}\tilde{x} \to T_t\tilde{x}$ uniformly on $[0,\alpha]$. Therefore there exists a number $\varepsilon_1 > 0$ such that $\|T_t^{(\varepsilon)}\tilde{x}\| < \delta - 1$ on $[0,\alpha]$ for $0 < \varepsilon \leq \varepsilon_1$. Again by (H5) and Proposition 1.1 we obtain that $x^{(\varepsilon)}(t;T_\alpha^{(\varepsilon)}\tilde{x}) = x^{(\varepsilon)}(t+\alpha;\tilde{x})$ exists on $[0,\alpha]$ for $0 < \varepsilon \leq \varepsilon_1$ and $\|T_{\alpha+t}^{(\varepsilon)}\tilde{x}\| < \delta - 1/2$ on $[0,\alpha]$. Then we can chose a number $\varepsilon_2 \leq \varepsilon_1$ such that $\|T_t^{(\varepsilon)}\tilde{x}\| < \delta - 1$ on $[0,2\alpha]$ for $0 < \varepsilon \leq \varepsilon_2$. In a finite number of steps we arrive at a number $\varepsilon_0 > 0$ such that $T_t^{(\varepsilon)}\tilde{x}$ exists on $[0,T]$ with $\|T_t^{(\varepsilon)}\tilde{x}\| < \delta - 1$ on $[0,T]$ for all $\varepsilon\in(0,\varepsilon_0]$. The rest of the proof is quite analogous to that of Proposition 3.3.

Remark 4.1. If $[0,t^+_{\tilde{x},\varepsilon})$ denotes the maximal interval of existence of $x^{(\varepsilon)}(t;\tilde{x})$ then the proof of Proposition 4.2 shows that

$$\liminf_{\varepsilon \to 0+} t^+_{\tilde{x},\varepsilon} \geq t^+_{\tilde{x}}.$$

§ 6. Numerical approximation by averaging projections

The objective of this section is to study the approximation of solutions of the nonlinear functional-differential equation (2.1) by solutions of ordinary differential equations acting on a sequence of finite dimensional subspaces of X. We take the averaging projections of Banks and Burns [2] and deal with the case where (2.1) is the difference-differential equation

$$\dot{x}(t) = h(x(t),x(t-k_1),\ldots,x(t-k_m)),\ t \geq 0, \tag{5.1}$$

$0 = k_0 < k_1 < \ldots < k_m$. The assumptions on h are the same as stated in Remark 1.2. Equation (5.1) is the more difficult case compared with an equation where a condition of type (1.5) holds. Some technical difficulties which we couldn't overcome are the reason that we finally can get the desired approximation results only for initial data $\tilde{x} = (\varphi(0),\varphi)$, where φ is Lipschitz on [-r,0], i.e. $\varphi \in W^{1,\infty}(-r,0;\mathbb{R}^n)$. We denote the subspace of these initial data in X with $\mathcal{W}^{1,\infty}$. For the rest of this section we fix a solution $x(t;\tilde{x})$, $\tilde{x} \in \mathcal{W}^{1,\infty}$, of (5.1) which exists on [-r,T], T > 0. There exists a constant K such that $|x(t;\tilde{x})| \leq K$ on [-r,T]. We define the map $\tilde{h}(y_0,\ldots,y_m) = h(\bar{y}_0,\ldots,\bar{y}_m)$ where $\bar{y}_j = y_j$ for $|y_j| \leq K+\rho$ and $= \frac{K+\rho}{|y_j|} y_j$ for $|y_j| > K+\rho$. The positive number ρ will be chosen appropriately in the course of the proof. It is easily seen that $\tilde{h}$ is globally Lipschitz with Lipschitz constant $\tilde{L} = L(K+\rho)$. According to Remark 1.2 we can take $X = \mathbb{R}^n \times L^2(-r,0;\mathbb{R}^n)$ as state space for the equation

$$\dot{x}(t) = \tilde{h}(x(t),x(t-k_1),\ldots,x(t-k_m)) \tag{5.2}$$

By definition of $\tilde{h}$ it is clear that $x(t;\tilde{x})$ is also the

solution of (5.2) with initial data $\tilde{x}$ on $[-r,T]$.

The sequence (X^N) of subspaces of X is defined as in [2], i.e. a typical element of X^N is $(\eta, \sum_{j=1}^{N} v_j^N \chi_j^N)$, where $\eta, v_1^N, \dots, v_N^N$ are elements in $\mathbb{R}^n$ and χ_j^N is the characteristic function of the interval $[t_j^N, t_{j-1}^N)$, $j=1,\dots,N$, $t_j^N = -j\frac{r}{N}$.
The projections $\pi^N : X \to X^N$ are defined by $\pi^N(\eta,\varphi) = (\eta, \sum_{j=1}^{N} \varphi_j^N \chi_j^N)$ where

$$\varphi_j^N = \frac{N}{r} \int_{t_j}^{t_{j-1}^N} \varphi(s)ds \ , \ j = 1,\dots,N. \tag{5.3}$$

It is easy to see that π^N is a contraction on X for $N=1,2,\dots$.
Moreover,

$$\lim_{N\to\infty} \pi^N\tilde{x} = \tilde{x} \ , \ \tilde{x}\in X \tag{5.4}$$

If we define $\|(\eta,\varphi)\|_\infty = \max(|\eta|, \|\varphi\|_\infty)$ for $(\eta,\varphi)\in\mathbb{R}^n\times L^\infty(-r,0;\mathbb{R}^n)$ then an easy calculation shows

$$\|\pi^N(\eta,\varphi)\|_\infty \leq \|(\eta,\varphi)\|_\infty \tag{5.5}$$

for $(\eta,\varphi)\in\mathbb{R}^n\times L^\infty(-r,0;\mathbb{R}^n)$. Note, that $\|\pi^N(\eta,\varphi)\|_\infty = \max(|\eta|, |\varphi_1^N|, \dots, |\varphi_N^N|)$, where the φ_j^N are defined in (5.3).

Following[2] we define the operators $\tilde{A}_0^N : X^N \to X^N$ by

$$\tilde{A}_0^N(\eta, \sum_{j=1}^{N} v_j^N \chi_j^N) = (0, \frac{N}{r}\sum_{j=1}^{N} (v_{j-1}^N - v_j^N)\chi_j^N) \tag{5.6}$$

where $v_0^N = \eta$ and put

$$A_0^N\tilde{x} = \tilde{A}_0^N \pi^N\tilde{x} \ , \ \tilde{x}\in X \tag{5.7}$$

Obviously, A_0^N is a bounded linear operator with $\mathcal{D}(A_0^N) = X$ and therefore generates the linear C_0-semigroup

$S_t^N = e^{A_0^N t}$, $t \geq 0$.

We need some results on A_o^N and S_t^N with respect to the space X:

Lemma 5.1. a) The operator $A_o^N - \frac{1}{2}I$ is dissipative for all N and

$$\|S_t^N\tilde{x}\| \leq e^{1/2\,t}\|\tilde{x}\| \;,\; \tilde{x}\in X \;,\; t \geq 0.$$

b) For all $\tilde{x}\in X$

$$\lim_{N\to\infty} S_t^N\tilde{x} = S_t\tilde{x}$$

uniformly on compact intervals.

Proof. In order to prove part a) we first take
$\tilde{x} = (\eta, \sum_{j=1}^{N} v_j^N x_j^N)\in X^N$. From (3.3) we get (note, that the duality map is the identity map in case p=2)

$$\langle A_o^N\tilde{x},\tilde{x}\rangle = \sum_{j=1}^{N} (v_{j-1}^N - v_j^N)v_j^N =$$

$$= -\frac{1}{2}\sum_{j=1}^{N} |v_{j-1}^N - v_j^N|^2 + \frac{1}{2}|\eta|^2 - \frac{1}{2}|v_N^N|^2$$

$$\leq \frac{1}{2}|\eta|^2 \leq \frac{1}{2}\|\tilde{x}\|^2.$$

For general $\tilde{x} = (\eta,\varphi)\in X$ we write $\tilde{x} = \pi^N\tilde{x}+\tilde{y}^N$ where $\pi^N\tilde{y}=0$ and therefore $A_o^N\tilde{y}^N = 0$ according to (5.7). This gives

$$\langle A_o^N\tilde{x},\tilde{x}\rangle = (A_o^N\pi^N\tilde{x},\pi^N\tilde{x}) \leq \frac{1}{2}|\eta|^2 \leq \frac{1}{2}\|\tilde{x}\|^2.$$

The estimate on S_t^N is clear by the dissipativeness of $A_o^N-\frac{1}{2}I$.
Part b) is clear by standard results of linear semigroup theory (cf. for instance [15]). We refer to [2] for a proof of $A_o^N\tilde{x} \to A_o\tilde{x}$ for $\tilde{x}\in\mathcal{D}$.

Remark 5.1. If we choose the columns of the matrix $(I, I\chi_1^N(s),\dots,I\chi_N^N(s))$ as a base of X^N then $\tilde{A}_o^N$ has the following matrix representation $[\tilde{A}_o^N]$:

$$[\tilde{A}_o^N] = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \frac{N}{r}I & -\frac{N}{r}I & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \\ 0 & \cdots\; 0 & \frac{N}{r}I & -\frac{N}{r}I \end{pmatrix}$$

For $\tilde{x} = (\eta, \sum_{j=1}^{N} v_j^N x_j^N) \in X^n$ we denote with $w^N(t)$ the coordinate vector of $S_t^N \tilde{x}$ with respect to the base chosen above. Then $w^N(t)$ is the solution of

$$\dot{w}^N = [\tilde{A}_o^N] w^N$$

with $w^N(0) = (\eta, v_1^N, \ldots, v_N^N)$. For $\tilde{x} = \pi^N \tilde{x} + \tilde{y}^N$ we get $S_t^N \tilde{x} = S_t^N \pi^N \tilde{x} + S_t^N \tilde{y}^N$, where $S_t^N \tilde{y}^N \equiv \tilde{y}^N$. By (5.4) and part b) of Lemma 5.1 we get $\tilde{y}^N \to 0$ and

$$\lim_{t \to \infty} S_t^N \pi^N \tilde{x} = S_t \tilde{x}\ , \quad \tilde{x} \in X,$$

uniformly on compact intervals.

Remark 5.2. Since $[\tilde{A}_o^N]$ has a special structure it is not difficult to obtain an explicit representation of $[e^{\tilde{A}_o^N t}]$:

$$[e^{\tilde{A}_o^N t}] = \begin{pmatrix} 1 & 0 \\ \exp(\frac{tN}{r} B^N) - I)(B^N)^{-1} a^N & \exp(\frac{tN}{r} B^N) \end{pmatrix} \qquad (5.8)$$

where $a^N = \mathrm{col}\ (I, 0, \ldots, 0)$ and

$$B^N = \begin{pmatrix} -I & 0 & \cdots & 0 \\ I & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & & & \\ 0 & \cdots\; 0 & I & -I \end{pmatrix}.$$

We need the explicit representation of $[e^{\tilde{A}_o^N t}]$ in order to prove

Lemma 5.2. S_t^N , $t \geq 0$, is a uniformly equicontinuous semigroup on $\mathbb{R}^n \times L^\infty(-r,0;\mathbb{R}^n)$, i.e.

$$\|S_t^N \tilde{x}\|_\infty \geq 3\|\tilde{x}\|_\infty , \quad \tilde{x} \in \mathbb{R}^n \times L^\infty(-r,0;\mathbb{R}^n),$$

for $t \geq 0$, $N = 1,2,\ldots.$

Proof. Since $S_t^N \tilde{x} = S_t^N \pi^N \tilde{x} + \tilde{y}^N$, by Remark 5.1, and therefore $\|S_t^N \tilde{x}\|_\infty \leq \|S_t^N \pi^N \tilde{x}\|_\infty + \|\tilde{y}^N\|_\infty \leq \|S_t^N \pi^N \tilde{x}\|_\infty + \|\tilde{x}\|_\infty$, we only need to prove that S_t^N is uniformly equicontinuous on X^N with respect to the sup-norm. For $\tilde{x} = (\eta, \sum_{j=1}^{N} v_j^N \chi_j^N)$ we get

$$[a^{\tilde{A}_o^N t}] \operatorname{col}(\eta, v_1^N, \ldots, v_N^N) =$$

$$[e^{\tilde{A}_o^N t}] \operatorname{col}(\eta, 0, \ldots, 0) + [e^{\tilde{A}_o^N t}] \operatorname{col}(0, v_1^N, \ldots, v_N^N).$$

From the representation of $\exp[\tilde{A}_o^N t]$ given in Remark 5.2 we get

$$\begin{aligned} a &= [e^{\tilde{A}_o^N t}]\operatorname{col}(\eta,0,\ldots,0) \\ &= \operatorname{col}\Big(\eta, \big(1-e^{-\frac{Nt}{r}}\big(1+\tfrac{Nt}{r}\big)\big)\eta, \ldots, \big(1-e^{-\frac{Nt}{r}} \sum_{j=0}^{N-1} \tfrac{1}{j!}\big(\tfrac{Nt}{r}\big)^j\big)\eta\Big) \end{aligned}$$

and

$$\begin{aligned} b &= [e^{\tilde{A}_o^N t}]\operatorname{col}(0,v_1^N,\ldots,v_N^N) \\ &= \operatorname{col}\Big(0, v_1^N e^{-\frac{Nt}{r}}, \ldots, e^{-\frac{Nt}{r}} \sum_{j=o}^{N-1} \tfrac{1}{j!} v_{N-j}^N \big(\tfrac{Nt}{r}\big)^j\Big). \end{aligned}$$

From these explicit representations we immediately get

$$\|a\|_\infty \leq |\eta|, \quad \|b\|_\infty \leq \max_j |v_j^N|$$

which proves

$$\|[e^{\tilde{A}_o^N t}]\operatorname{col}(\eta, v_1^N, \ldots, v_N^N)\|_\infty \leq 2\|\tilde{x}\|_\infty .$$

For $\tilde{x} = (\eta,\varphi) \in X$ we define

$$\tilde{f}_N(\tilde{x}) = \tilde{h}(\eta, \varphi^N(-k_1),\ldots,\varphi^N(-k_m))$$

and

$$f_N(\tilde{x}) = h(\eta,\varphi^N(-k_1),\ldots,\varphi^N(-k_m))$$

where $(\eta,\varphi^N) = \pi^N(\eta,\varphi)$. Note, that $\tilde{f}_N$ and f_N are well-defined on X since φ^N is right-hand continuous on $[-r,0)$. The corresponding definition in the general case of equation (2.1) is $f_N = f\circ\pi^N$ which makes sense if $X^N \subset \mathcal{D}(f)$ for all N.

As an immediatly consequence of (5.5) and the Lipschitz-continuity of h and $\tilde{h}$ we get

$$|f_N(\tilde{x}) - f_N(\tilde{y})| \leq L(M)\|\tilde{x}-\tilde{y}\|_\infty$$

for $\tilde{x},\tilde{y} \in \mathbb{R}^n \times L^\infty(-r,0;\mathbb{R}^n)$ with $\|\tilde{x}\|_\infty \leq M$, $\|\tilde{y}\|_\infty \leq M$ and

$$|\tilde{f}_N(\tilde{x}) - \tilde{f}_N(\tilde{y})| \leq \tilde{L}\|\tilde{x}-\tilde{y}\|_\infty \tag{5.9}$$

for $\tilde{x},\tilde{y} \in \mathbb{R}^n \times L^\infty(-r,0;\mathbb{R}^n)$.

Since $\|\pi^N\tilde{x} - \pi^N\tilde{y}\|_\infty \leq (\frac{N}{r})^{1/2} \|\pi^N\tilde{x}-\pi^N\tilde{y}\| \leq (\frac{N}{r})^{1/2} \|\tilde{x}-\tilde{y}\|$, we also get

$$|\tilde{f}_N(\tilde{x})-\tilde{f}_N(\tilde{y})| \leq (\frac{N}{r})^{1/2} \tilde{L}\|\tilde{x}-\tilde{y}\| \tag{5.10}$$

for $\tilde{x},\tilde{y} \in X$.

In order to make the representation not to complicated we only deal with $\tilde{f}_N$ in detail and give the modifications for f_N in a remark. Corresponding to $\tilde{f}_N$ we define the nonlinear operators

$$A_1^N\tilde{x} = (\tilde{f}_N(\tilde{x}),0) \quad , \ \tilde{x}\in X$$

and $A^N = A_o^N + A_1^N$.

Lemma 5.3. a) $A^N - \omega I$ is dissipative with $\omega = \frac{1}{2}+(\frac{N}{r})^{1/2}\tilde{L}$.

b) A^N is the infinitesimal generator of a strongly continuous semigroup T_t^N , $t \geq 0$, such that

$$\|T_t^N\tilde{x} - T_t^N\tilde{y}\| \leq e^{\omega t}\|\tilde{x}-\tilde{y}\| \quad , \ \tilde{x},\tilde{y}\in X \ , \ t \geq 0.$$

Moreover, $T_t^N\tilde{x}$ is the unique solution of the integral equation

$$T_t^N\tilde{x} = S_t^N\tilde{x} + \int_o^t S_{t-s}^N A_1^N T_S^N\tilde{x}ds \ , \ t \geq 0. \tag{5.11}$$

Proof. Part a) follows easily from Lemma 5.1, a) and (5.10). Part b) is proved by an application of the Crandall-Liggett theorem [9] or follows from [18] where also the result regarding the integral equation (5.11) is proved. If one applies the Crandall-Liggett theorem one has to prove $\mathcal{R}(I-\lambda A^N) = X$ for $\lambda\in(0,\frac{1}{\omega})$. But this is easily done by a discussion of the fixed point equation

$$\tilde{y} = (I-\lambda A_o^N)^{-1}\tilde{x} + \lambda(I-\lambda A_o^N)^{-1}A_1^N\tilde{y} \ , \ \tilde{x},\tilde{y}\in X.$$

Remark 5.3. We choose the same base of X^N as in Remark 5.1 and define

$$\tilde{F}_N(\eta,v_1^N,\ldots,v_N^N) = \mathrm{col}(\tilde{f}_N(\eta, \sum_{j=1}^N v_j^N x_j^N), 0,\ldots,0).$$

For $\tilde{x} = (\eta, \sum_{j=1}^N v_j^N x_j^N)\in X^N$ let $w^N(t)$ denote the coordinate vector of $T_t^N\tilde{x}$. Then $w^N(t;x)$ is the solution of

$$\dot{w}^N = [\tilde{A}_o^N]w^N + \tilde{F}_N(w^N) \ , \quad t \geq 0 \ , \tag{5.12}$$

$w^N(0) = \mathrm{col}(\eta,v_1^N,\ldots,v_N^N)$. This follows from the integral equation (5.11) and the fact that X^N is invariant with respect to A^N. For general $\tilde{x}\in X$ we write $\tilde{x} = \pi^N\tilde{x}+\tilde{y}^N$ and get

$$\|T_t^N\tilde{x} - T_t^N\pi^N\tilde{x}\| \leq e^{\omega t}\|\tilde{x}-\pi^N\tilde{x}\| \ , \ t \geq 0,$$

i.e. by (5.4) we have $\lim\limits_{N\to\infty} \|T_t^N\tilde{x} - T_t^N\pi^N\tilde{x}\| = 0$
uniformly on compact intervals.

Corresponding to equation (5.2) for $\varepsilon > 0$ we define f_ε as in Section 3 and get from Proposition 3.3

$$\lim_{\varepsilon\to 0+} T_t^{(\varepsilon)}\tilde{x} = T_t\tilde{x} \ , \ \tilde{x}\in X \tag{5.13}$$

uniformly on compact intervals, where $T_t^{(\varepsilon)}$, $t \geq 0$, is the solution semigroup of equation (3.6) with infinitesimal generator $A_\varepsilon = A_o + A_{1,\varepsilon}$, $A_{1,\varepsilon}\tilde{x} = (f_\varepsilon(\tilde{x}),0)$. The calculations given in Remark 1.2 show that the function $\gamma_\beta(t)$ in (H2) can be chosen independently of β , $\gamma(t) = (m+1)\tilde{L}\, t^{1/2}$. This function can also be used for all equation (3.6) with $\varepsilon\in(0,k_1]$. Therefore $A_\varepsilon - \omega I$ is dissipative with $\omega=\frac{1}{2}+\frac{1}{\varepsilon}\gamma(\varepsilon) = \frac{1}{2}+(m+1)\tilde{L}\varepsilon^{-1/2}$ by Proposition 3.2, $\varepsilon \leq \min(k_1,1)$.
In the next step we define for $\varepsilon > 0$

$$f_{N,\varepsilon}(\tilde{x}) = \frac{1}{\varepsilon}\int_0^\varepsilon \tilde{f}_N(S_\tau\tilde{x})d\tau \quad , \ \tilde{x}\in X,$$

and the operator $A_\varepsilon^N = A_o^N + A_{1,\varepsilon}^N$, where $A_{1,\varepsilon}^N\tilde{x} = (f_{N,\varepsilon}(\tilde{x}),0)$ for $\tilde{x}\in X$.

<u>Lemma 5.4.</u> a) $A_\varepsilon^N - \omega I$ is dissipative with $\omega = \frac{1}{2} + (m+1)\,\tilde{L}\varepsilon^{-1/2}$ for all $N > N_o$ and $A_\varepsilon^N\tilde{x} \to A_\varepsilon\tilde{x}$ for $\tilde{x}\in$.

b) A_ε^N is the infinitesimal generator of a nonlinear semigroup $T_t^{(\varepsilon,N)}$, $t \geq 0$, such that

$$\|T_t^{(\varepsilon,N)}\tilde{x} - T_t^{(\varepsilon,N)}\tilde{y}\| \leq e^{\omega t}\|\tilde{x}-\tilde{y}\|$$

for $t \geq 0$ $\tilde{x},\tilde{y}\in X$ and N _ N_o.

c) $T_t^{(\varepsilon,N)}x$ is the unique solution of the integral equation

$$T_t^{(\varepsilon,N)}\tilde{x} = S_t^N\tilde{x} + \int_0^t S_{t-s}^N\ A_{1,\varepsilon}^N T_s^{(\varepsilon,N)}\tilde{x}ds \ , \ t \geq 0 \tag{5.14}$$

d) For all $\varepsilon > 0$ we have

$$\lim_{N\to\infty} T_t^{(\varepsilon,N)}\tilde{x} = T_t^{(\varepsilon)}\tilde{x} \ , \ \tilde{x}\in X,$$

uniformly on compact intervals.

<u>Proof</u>. We first prove that $f_{N,\varepsilon}$ is globally Lipschitz on X with Lipschitz constant $(m+1)\tilde{L}\varepsilon^{-1/2}$, which proves that $A_\varepsilon^N - \omega I$ is dissipative.

For $\tilde{x}_i = (\eta_i,\varphi_i)\in X$, $i = 1,2$, define the function

$$\psi_i(s) = \begin{cases} \varphi_i(s) & \text{on } [-r,0), \\ \eta_i & \text{on } [0,\infty) , \end{cases} \qquad , \ i = 1,2 .$$

Then

$$|f_{N,\varepsilon}(\tilde{x}_1)-f_{N,\varepsilon}(\tilde{x}_2)| \leq \frac{1}{\varepsilon}\int_0^\varepsilon |\mathcal{F}_N(\eta_1,(\psi_1)_\tau)-\mathcal{F}_N(\eta_2,(\psi_2)_\tau)|\,d\tau$$

$$= \frac{1}{\varepsilon}\int_0^\varepsilon \Big|\tilde{h}\Big(\eta_1,\frac{N}{r}\int_{t_{j_1}^N}^{t_{j_1-1}^N} \varphi_1(s+\tau)ds,\dots,\frac{N}{r}\int_{t_{j_m}^N}^{t_{j_m-1}^N} \varphi_1(s+\tau)ds\Big) -$$

$$- \tilde{h}\Big(\eta_2,\dots,\frac{N}{r}\int_{t_{j_m}^N}^{t_{j_m-1}^N} \varphi_2(s+\tau)ds\Big)\Big|\,d\tau$$

where we choose N such that $\frac{r}{N} < \min_j(k_{j+1}- k_j)$ and j_μ is such that $k_\mu\in[t_{j_\mu}^N,t_{j_\mu-1}^N)$. Since $\tilde{h}$ is globally Lipschitz we get (for $\varepsilon\leq 1$)

$$|f_{N,\varepsilon}(\tilde{x}_1) - f_{N,\varepsilon}(\tilde{x}_2)| \leq \frac{\tilde{L}}{\varepsilon}\int_0^\varepsilon\Big(|\eta_1-\eta_2| + \frac{N}{r}\sum_{\mu=1}^m \int_{t_{j_\mu}^N}^{t_{j_\mu-1}^N} |\varphi_1(\tau+s)-\varphi_2(\tau+s)|ds\Big)d\tau$$

$$\leq \frac{\tilde{L}}{\varepsilon}\Big(\frac{N}{r}\int_{t_1^N}^{0}\int_0^\varepsilon |\eta_1-\eta_2|\,ds\,d\tau + \frac{N}{r}\sum_{\mu=1}^m\int_{t_{j_\mu}^N}^{t_{j_\mu-1}^N}\int_0^\varepsilon |\psi_1(\tau+s)-\psi_2(\tau+s)|\,ds\,d\tau\Big)$$

$$\leq \frac{\tilde{L}}{\varepsilon}\cdot\frac{N}{r}\varepsilon^{1/2}\left(\int_{t_1^N}^{0}\left(\int_{-r}^{1}|\psi_1(s)-\psi_2(s)|^2 ds\right)^{1/2} d\tau + \sum_{\mu=1}^{m}\int_{t_{j_\mu}^N}^{t_{j_\mu-1}^N}\left(\int_{-r}^{0}|\psi_1(s)-\psi_2(s)|^2 ds\right)^{1/2} d\tau\right)$$

$$\leq \tilde{L}\varepsilon^{-1/2}(m+1)\|\tilde{x}_1-\tilde{x}_2\|.$$

In order to prove the second claim of part a) for $\tilde{x}=(\varphi(0),\varphi)\in\mathcal{D}$ we consider the estimate

$$|f_{N,\varepsilon}(\tilde{x})-f_\varepsilon(\tilde{x})| \leq \frac{1}{\varepsilon}\int_0^\varepsilon |\tilde{f}_N(S_\tau\tilde{x})-f(S_\tau\tilde{x})|d\tau$$

$$= \frac{1}{\varepsilon}\int_0^\varepsilon |\tilde{h}(\varphi(0),\frac{N}{r}\int_{t_{j_1}^N}^{t_{j_1-1}^N}\varphi(\tau+s)ds,\ldots)-\tilde{h}(\varphi(0),\varphi(\tau-k_1),\ldots)|d\tau$$

$$\leq \frac{\tilde{L}}{\varepsilon}\int_0^\varepsilon \frac{N}{r}\left(\sum_{\mu=1}^{m}\int_{t_{j_\mu}^N}^{t_{j_\mu-1}^N}|\varphi(\tau+s)-\varphi(\tau-k_\mu)|ds\right)d\tau$$

where we again choose N such that

$$\frac{r}{N} < \min_j(k_{j+1}-k_j).$$

Since φ is uniformly continuous on $[-r,0]$ for arbitrary $\mu > 0$ we can define N_0 such that for $N \geq N_0$ we have

$$|\varphi(\tau+s)-\varphi(\tau-k_\mu)| \leq \mu \quad \text{for } s\in[t_j^N,t_{j-1}^N].$$

From the estimate given above we therefore obtain (note, that $k_\mu\in[t_{j_\mu}^N,t_{j_\mu-1}^N)$) for $N \geq N_0$

$$|f_{N,\varepsilon}(\tilde{x})-f_\varepsilon(\tilde{x})| \leq \frac{\tilde{L}}{\varepsilon}\,\frac{N}{r}\cdot\frac{r}{N}\,\mu\varepsilon = \tilde{L}\mu$$

which proves $A_{1,\varepsilon}\tilde{x} \to A_{1,\varepsilon}\tilde{x}$ for $\tilde{x}\in\mathcal{D}$ as $N \to \infty$.

The proof of parts b) and c) is quite analogous to the proof given for Lemma 5.3.
Part d) follows from Corollary 4.2 of [8] using part a) of the lemma.

Lemma 5.5. For each $\tilde{x}\in\mathring{W}^{1,\infty}$ we have

$$\lim_{\varepsilon\to 0+} \|T_t^{(\varepsilon,N)}\tilde{x} - T_t^N\tilde{x}\|_\infty = 0$$

uniformly with respect to N and uniformly on compact intervals.

Proof. From (5.11) and (5.14) we get

$$\|T_t^{(\varepsilon,N)}\tilde{x} - T_t^N\tilde{x}\|_\infty \le \int_0^t \|S_{t-s}^N(f_{N,\varepsilon}(T_s^{(\varepsilon,N)}\tilde{x}) - \tilde{f}_N(T_s^N\tilde{x}),0)\|_\infty ds$$

$$\le 3\int_0^t |f_{N,\varepsilon}(T_s^{(\varepsilon,N)}\tilde{x}) - \tilde{f}_N(T_s^N\tilde{x})|ds$$

$$\le 3\int_0^t |f_{N,\varepsilon}(T_s^{(\varepsilon,N)}\tilde{x}) - f_{N,\varepsilon}(T_s^N\tilde{x})|ds$$

$$+ 3\int_0^t |f_{N,\varepsilon}(T_s^N\tilde{x}) - \tilde{f}_N(T_s^N\tilde{x})|ds$$

$$\le 3L\int_0^t \|T_s^{(\varepsilon,N)}\tilde{x} - T_s^N\tilde{x}\|_\infty ds + 3\int_0^t |f_{N,\varepsilon}(T_s^N\tilde{x}) - \tilde{f}_N(T_s^N\tilde{x})|ds. \tag{5.15}$$

Here we have used the fact that from (5.9) we immediately get the same condition for all $f_{N,\varepsilon}$. For the second integral on the right-hand side of the estimate we get

$$\int_0^t |f_{N,\varepsilon}(T_s^N\tilde{x}) - \tilde{f}_N(T_s^N\tilde{x})|ds$$

$$\le \frac{L}{\varepsilon}\int_0^t\int_0^\varepsilon \|\pi^N S_\tau T_s^N\tilde{x} - T_s^N\tilde{x}\|_\infty d\tau ds. \tag{5.16}$$

Using for fixed τ the representations $\pi^N S_\tau T_s^N\tilde{x} = (\eta(s), \sum_{j=1}^N \psi_j^N(s)\chi_j^N)$ and $T_s^N\tilde{x} = (\eta(s), \sum_{j=1}^N \varphi_j^N(s)\chi_j^N)$ we get

$$\|\pi^N S_\tau T_s^N\tilde{x} - T_s^N\tilde{x}\|_\infty = \max_j |\varphi_j^N(s) - \psi_j^N(s)| .$$

Recalling the definition of S_τ we get

$$\begin{aligned}\psi_j^N - \varphi_j^N &= \frac{N}{r}(t_{j-\kappa}^N - (t_j^N+\tau))(\varphi_{j-\kappa+1}^N - \varphi_j^N) \\ &+ \frac{N}{r}(t_{j-1}^N+\tau - t_{j-\kappa}^N)(\varphi_{j-\kappa}^N - \varphi_j^N)\end{aligned} \tag{5.17}$$

where κ is defined by $t_j^N+\tau\in[t_{j-\kappa+1}^N, t_{j-\kappa}^N)$ and we have dropped the argument s.
For any natural number we have

$$|\varphi_{j-\rho}^N - \varphi_j^N| \le \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N} |\varphi(s+\rho\frac{r}{N})-\varphi(s)|ds$$

$$\le \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N}\int_s^{s+\rho\frac{r}{N}} |\dot\varphi(\sigma)|d\sigma\, ds \le \rho\frac{r}{N}\|\dot\varphi\|_\infty$$

Therefore we get from (5.17)

$$|\psi_j^N-\varphi_j^N| \le [(t_{j-\kappa}^N-(t_j^N+\tau))(\kappa-1) + (t_{j-1}^N+\tau-t_{j-\kappa}^N)\kappa]\,\|\dot\varphi\|_\infty = \tau\,\|\dot\varphi\|_\infty$$

This estimate together with (5.16) shows

$$\int_0^t |f_{N,\varepsilon}(T_s^N\tilde x) - f_N(T_s^N\tilde x)|ds \le \tilde L\varepsilon t\|\dot\varphi\|_\infty$$

Using this in (5.15) an application fo Gronwal's inequality gives

$$\|T_t^{(\varepsilon,N)}\tilde x - T_t^N\tilde x\|_\infty \le 3\tilde L\,\varepsilon\, t\;\|\dot\varphi\|_\infty\exp(3\tilde L t)$$

which proves the Lemma.

We now are in the position to prove the main result of this section. Define F_N corresponding to f_N as $\tilde F_N$ was defined in Remark 5.3 corresponding to $\tilde f_N$.

Proposition 5.1. For $\tilde x\in\mathcal{W}^{1,\infty}$ let $u^N(t)$ be the solution of

$$\dot u = [\tilde A_o^N]u + F_N(u)\;,\; t\ge 0\;, \tag{5.18}$$

with $u(0) = col(\eta, v_1^N,\dots,v_N^N)$, $\pi^N\tilde x = (\eta, \sum_{j=1}^N v_j^N x_j^N)$, and define

$$\tilde x^N(t) = (u_o^N(t),\; \sum_{j=1}^N u_j^N(t)x_j^N),\quad t\ge 0.$$

Then

$$\lim_{N\to\infty} \tilde{x}^N(t) = T_t\tilde{x} \text{ ,}$$

uniformly on compact intervals.

Proof. We first deal with the global case, i.e. with equations (5.2) and (5.12).
For $\mu > 0$ according to Proposition 3.3 and Lemma 5.5 there exists an $\varepsilon_0 > 0$ such that

$$\|T_t\tilde{x} - T_t^{(\varepsilon_0)}\tilde{x}\| < \mu \text{ ,}$$

$$\|T_t^{(\varepsilon_0,N)}\tilde{x} - T_t^N\tilde{x}\| < \mu$$

for $t\in[0,T]$ and all N. Then we can choose N_0 such that for $N \geq N_0$ and $t\in[0,T]$ (cf. Lemma 5.4, d))

$$\|T_t^{\varepsilon_0}\tilde{x} - T_t^{(\varepsilon_0,N)}\tilde{x}\| < \mu$$

The estimates together give

$$\|T_t\tilde{x} - T_t^N x\| \leq 3\mu$$

for all $N \geq N_0$ and $t\in[0,T]$. This and with Remark 5.3 proves the result for the global case.

If we consider the local case we choose the constant ρ which appears in the definition of $\tilde{h}$ equal to 1 and get

$$T_t\tilde{x} = \lim_{N\to\infty} (w_0^N(t), \sum_{j=1}^{N} w_j^N(t)x_j^N) \tag{5.19}$$

uniformly on compact intervals, where $w^N(t)$ is the solution of equation (5.12) with

$w^N(0) = \mathrm{col}(\varphi(0),\varphi_j^N,\ldots,\varphi_N^N)$, $(\varphi(0), \sum_{j=1}^{N} \varphi_j^N x_j^N) = \pi^N\tilde{x}$.

we only need to show that for N sufficiently large we have

$\|w^N(t)\|_\infty \leq K+1$ on $[0,T]$. Then $w^N(t) = u^N(t)$ for N sufficiently large, because $\tilde{F}_N(u) = F_N(u)$ for $\|u\|_\infty \leq K+1$.

Using Lemma 5.2 and (5.9) we get from

$$w^N(t) = [e^{\tilde{A}_0^N t}]w^N(0) + \int_0^t [e^{\tilde{A}_0^N(t-s)}]\ \tilde{F}_N(w^N(s))ds \tag{5.20}$$

by an application of Gronwal's inequality that there exists a constant P

$$\|w^N(t)\|_\infty \leq P$$

on $[0,T]$ for all N.

Differentiating (5.20) we get

$$\dot{w}^N(t) = [e^{\tilde{A}_0^N t}][\tilde{A}_0^N]\ w^N(0) + \int_0^t [e^{\tilde{A}_0^N(t-s)}][\tilde{A}_0^N]\tilde{F}_N(w^N(s))ds$$
$$+ \tilde{F}_N(w^N(t)) \tag{5.21}$$

An easy calculation shows that

$$\|[\tilde{A}_0^N]w^N(0)\|_\infty = \|A_0^N\tilde{x}\|_\infty \leq P_1 \quad \text{for all } N$$

where $P_1 = \|\dot{\varphi}\|_\infty$ (note, $\tilde{x} = (\varphi(0),\varphi)$, $\varphi\epsilon\mathcal{W}^{1,\infty}$).

Therefore according to Lemma 5.2

$$\|[e^{\tilde{A}_0^N t}][\tilde{A}_0^N]\ w^N(0)\|_\infty \leq 3P_1$$

If we write the second term on the right-hand side of (5.21) explicitly we get

$$\frac{N}{r}\int_0^t \mathrm{col}(0,w^{-\frac{N}{r}(t-s)}\tilde{F}_N(w^N(s)),\ldots,\frac{1}{(N-1)!}(\frac{(t-s)N}{r})^{N-1}e^{-\frac{N}{r}(t-s)}\tilde{F}_N(w^N(s)))ds\,. \tag{5.22}$$

Using the uniform Lipschitz condition on $\tilde{F}_N$ and the boundednes of $\|w^N(t)\|_\infty$ we after some elementary calculation see that there exists a constant P_2 such that the sup-norms of (5.22) is uniformly bounded by P_2. It is clear that the third term on the

right-hand side of (5.21) is also uniformly bounded. Therefore, there exist a constant Q such that

$$\|\dot{w}^N(t)\|_\infty \leq Q$$

on [0,T] for all N. From this we get

$$|w_{j-1}^N(t) - w_j^N(t)| = \frac{r}{N}|\dot{w}_j^N(t)| \leq \frac{r}{N}Q$$

and

$$|w_{j+1}^N(t) - w_j^N(t)| = \frac{r}{N}|\dot{w}_{j+1}^N(t)| \leq \frac{r}{N}Q$$

for all N and $t\in[0,T]$.
For any natural number ρ we therefore have

$$\begin{aligned} &|w_{j-\rho}^N(t)-w_j^N(t)| \leq \rho\frac{r}{N}Q, \\ &|w_{j+\rho}^N(t) - w_j^N(t)| \leq \rho\frac{r}{N}Q \end{aligned} \tag{5.23}$$

for all N and $t\in[0,T]$.
Suppose that for $t\in[0,T]$ there exists a subsequence (N_k) which for simplicity we again denote by N such that

$$|w_{j_N}^N(t)| > K+1 \tag{5.24}$$

for at least one j_N , $1 \leq j_N \leq N$. By (5.23) and (5.24) we get

$$|w_{j_N\pm\rho}^N(t)| > K+\frac{1}{2} \tag{5.25}$$

for all k and all $\rho \leq [\frac{N}{2rQ}]$ ($[\alpha]$ denotes the largest integer $\leq\alpha$). If N_k denotes the smallest integer such that $k = [\frac{1}{2rQ}N_k]$ then $N_k \leq kN_1$ and $[\frac{1}{2rQ}N_k] \geq k$. Therefore from (5.25) we infer that

$$\left|\sum_{j=1}^{N_k} w_j^{N_k}(t)\chi_j^{N_k}(s)\right| > K+\frac{1}{2} \quad , \; k = 1,2,\ldots \tag{5.26}$$

for s in an intervall of length greater than $k\frac{r}{N_1} = \frac{r}{N_1}$.

In this case (5.26) shows that we cannot have

$$\| \sum_{j=1}^{N_k} w_j^{N_k}(t) x_j^{N_k} - x_t(t;\tilde{x}) \|_2 \to 0$$

as $k \to \infty$. This contradicts (5.19) and therefore $\|w^N(t)\|_\infty \leq K+1$ for all N and $t \in [0,T]$.

Remark 5.4. The proof given for Proposition 3.1 in the local case can easily be adapted to show that for $\tilde{x} \in \mathcal{W}^{1,\infty}$ we in fact have

$$\lim_{N\to\infty} \|\tilde{x}^N(t) - T_t \tilde{x}\|_\infty = 0$$

uniformly on compact intervals.

Remark 5.5. It is clear that the results of this paper cover also the case $\mathcal{C}_M \subset \mathcal{D}(f) \subset M \times L^p(-r,0;M)$ where M is an open subset of $\mathbb{R}^n$ and $\mathcal{C}_M = \{(\varphi(0),\varphi) \mid \varphi \in C(-r,0;M)\}$.

Remark 5.6. The approximation scheme given in Proposition 5.1 has been tested for various nonlinear examples. In each case the numerical results indicate linear convergence $u_0^N(t) \to x(t;\tilde{x})$ as $N \to \infty$.

References

[1] Banks,H.T.: Approximation methods for optimal control problems with delay-differential systems, to appear in Sêminaires IRIA: Analyse et contrôle de Systêmes(1976)

[2] Banks,H.T., and J.A.Burns: Hereditary control problems: numerical methods based on averaging approximations, Submitted to SIAM J. Control and Optimization (January 1976)

[3] Banks,H.T., J.A.Burns, E.M. Cliff and P.R.Thrift: Numerical solutions of hereditary control problems via an approximation technique, Brown University LCDS Techn.Rep. 75-6, Providence, R.I., October 1975.

[4] Banks,H.T, and A.Manitius: Projection series for retarded functional differential equations with applications to optimal control problems, J.differential Eqs. 18(1975), 296-332.

[5] Barbu,V.: Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden 1976.

[6] Borisovič,J.G., and A.S.Turbabin: On the Cauchy problem for linear nonhomogeneous differential equation with retarded argument, Soviet Math. Dokl. 10 (1969), 401-405.

[7] Brezis,H.: Opérateurs Maximaux Monotones et Semi-groupes de contractions dan les Espaces de Hilbert, North Holland, Amsterdam 1973

[8] Brezis,H., and A.Pazy: Convergence and approximation of semigroups of nonlinear operators in Banach spaces, J.Functional Analysis 9(1972), 63-74

[9] Crandall,M.G., and T.M.Liggett: Generation of semigroups of nonlinear transformations on general Banach spaces, Amer.J.Math.93 (1971), 265-298

[10] DeFigueiredo,D.G., and L.A.Karlovitz: On the radial projection in normed spaces, Bull.Amer.Math.Soc. 73 (1967),364-368.

[11] Delfour,M.C., and S.K.Mitter: Controllability, observability and optimal feedback control of affine hereditary differential systems, SIAM J. Control 10 (1972), 298-328.

[12] Halanay,A., and J.A.Yorke: Some new results and problems in the theory of differential-delay equations, SIAM Review 13 (1971), 55-80

[13] Kappel,F.: Differenzen-Differentialgleichungen, 5. Steiermärkisches mathematisches Symposium, Stift Rein 24.-27.Sept.1973, Ber.math.-stat.Sekt. Forschungszentrum Graz, No.1(1973),pp.77.

[14] Kato,T.: Non-linear semigroups and evolution equations, J.Math.Soc.Japan 19 (1967), 508-520.

[15] Pazy,A.: Semi-groups of linear operators and applications to partial differential equations, University of Maryland, Lecture Note No.10(1974).

[16] Plant,A.T.: Nonlinear semigroups of translations in Banach space generated by functional differential equation, to appear in J.math.Analysis Appl.

[17] Reddien,G.W., and G.F.Webb: Numerical approximation of nonlinear functional differential equations with L^2 initial functions, to appear.

[18] Webb,G.F.: Continuous nonlinear perturbations of linear accretive operators in Banach spaces, J. Functional Analysis 10(1972), 191-203

[19] Webb,G.F.: Autonomous nonlinear functional differential equation and nonlinear semigroups, J.math.Analysis Appl. 46 (1974), 1-12

[20] Webb,G.F.: Functional differential equations and non-linear semigroups in L^p-spaces, J.differential Eqs. 20 (1976), 71-89.

[21] Yosida,K.: Functional Analysis, 4th ed., Springer Verlag New York 1974.

B. Optimal Control of Volterra Integrodifferential Equations.

Summary.

In this paper we investigate the optimal control problem for Volterra integrodifferential equations

$$x(t) = A(t)x(t) + \int_{-\infty}^{t} F(t-s)x(s)ds + B(t)u(t) ,$$

where the target sets are elements of some function space,

$$x(t) = \varphi(t) \quad \text{for} \quad 0 < T-r \leq t \leq T .$$

The approach used is the abstract theory of Dubovitskii-Milyutin, the result is a necessary condition in form of a maximum principle.

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1. Introduction.

In this paper ~~we investigate~~ the optimal control problem for integrodifferential equations of Volterra type are investigated, where the target sets are elements of some function space. The approach used is the abstract theory of Dubovitskii and Milyutin, the result is a necessary condition in form of a maximum principle.

The form of the considered equation was motivated by the applications. A well known example for the occurance of integro-differential equations of Volterra type is the point kinetics model for nuclear reactors, given by the equations (see Akcasu et al. [1], DiPasquantonio and Kappel [21]):

$$\dot{p}(t) = -\sum_{j=1}^{N} \left(\frac{\beta_j}{l}\right)\left[p(t) - q_j(t)\right] - \frac{n_2}{l}\cdot\rho$$

$$\dot{q}_j(t) = \lambda_j\left[p(t) - q_j(t)\right] , \qquad j = 1,2,\ldots,N,$$

where p is the deviation of the reactor power from the equilibrium level n_2 , $q_j(t)$ gives the j-th group delayed neutron precursor density, l, β_j, λ_j , are positive constants and the reactivity $\rho = \rho(p_t)$ is a feedback-functional which may be taken as

$$\rho(p_t) = h\cdot p(t) + \int_{-\infty}^{t} f(t-s)p(s)ds + u(t) ,$$

h being some real constant.
Setting $x := \mathrm{col}(p,q_1,\ldots,q_N)$ and defining A, F, B, appropriate, this system can be written as

$$x(t) = Ax(t) + \int_{-\infty}^{t} F(t-s)x(s)ds + Bu(t) .$$

An important problem is to choose $u(t)$ such that the reactor is shut down with minimal Xenon^{135} build-up. The shut-down state corresponds to $x = \mathrm{col}(1,1,\ldots,1)$. In order to shut down the reactor we have to hold x on this value for some time interval.

Another problem is to choose $u(t)$ such that the level of the power equilibrium state is changed to another value in minimal time. In all these cases a certain terminal function has to be reached, while a certain cost functional has to take a minimal value.
There are some other fields where Volterra-integrodifferential equations occur, for instance viscoelasticity, theory of material with fading memory, population dynamics, spread of infectious deseases, prey predator problems, and control of chemical reactions.

We shall consider the equation

$$\dot{x}(t) = A(t)x(t) + \int_{-\infty}^{t} F(t-\tau)x(\tau)d\tau + B(t)u(t) \qquad (1)$$

with function space terminal conditions. This equation is sufficient general to cover the above mentioned applications.
In chapter 2. we formulate the problem and the main result of the paper. This is a maximum principle for the solution (x_o, u_o) of (1), which minimizes a certain cost functional.
In chapter 3. we sketch previous approaches to problems of such kind, which gave similar results by using other methods of proof. Chapters 4 to 11 present auxiliary material, which is needed in the proof of our theorem. We consider the "adjoint" equation in the space V of functions of bounded variation and prove the existence of an solution to this equation. In chapter 7. the theorem of Dubovitskii and Milyutin is cited in the used form. In chapter 8. we discuss the spaces that may be used and motivate our choice of the space $C^b(-\infty,\alpha]$ of continuous bounded functions. In chapter 9. we give an existence proof for solutions to (1) in the space C^b. Now we proceed to the application of the theorem of Dubovitskii and Milyutin. In chapter 1o. the cone of tangent directions to the solution set of (1) is given. Then in chapter 11. we determine the dual cone K to this tangent cone. Now the Euler equation of the problem can be established and is used in chapter 12. to derive the desired maximum condition. We conclude with some remarks concerning further necessary work.

2. The main result.

Our aim is to prove the following necessary condition for the solution of the below defined control problem (CP):

Theorem. Let $x_o(t)$, $u_o(t)$ be a solution of the control problem

$$(CP): \quad \dot{x}(t) = A(t)x(t) + \int_{-\infty}^{t} F(t-\tau)x(\tau)d\tau + B(t)u(t) \ ,$$

$$x(t) = \varphi(t) \quad \text{for} \quad -\infty < t \leq 0 \quad \text{and} \quad T-r \leq t \leq T \ ,$$

where $T-r > 0$,

$$\int_{0}^{T} \Phi(x(t),u(t),t)dt \rightarrow \min \ ,$$

where

1. A, B are $n \times n$ resp. $n \times r$ matrices, continuous and of bounded variation,

2. F(s) is of bounded variation,

$$\int_{+\infty}^{0} |F(s)|\, ds < \infty \ ,$$

3. φ is continuous and bounded on the intervals $-\infty < t \leq 0$, $T-r \leq t \leq T$.

Now let

4. $\Phi(x,u,t)$ be continuous in x,u, measurable in t, continuously differentiable with respect to x,u, and $\Phi_x(x,u,t)$, $\Phi_u(x,u,t)$ be bounded for bounded x,u,

5. M be a closed convex set in R^r, with $\mathring{M} \neq \emptyset$, $u(t) \in M$ for $0 \leq t \leq T$, $u \in L_\infty^{(r)}[0,T]$ (that is, bounded and measurable),

6. the condition of complete controllability of chapter 1o. for A, F, and B be fulfilled.

Then there exist a number $\lambda_o \geq 0$, a function $V(t)$ of bounded variation on $[0,T]$ and a function $\psi(t)$, solution

of

$$\psi(t) = \int_t^T (A^* \psi + \int_{+\infty}^{s} \psi(\tau) F(\tau - s) d\tau - \lambda_0 \Phi_x(x_0, u_0, s)) ds + V(t)$$

on $[0,T]$, such that λ_0 and ψ are not both zero and

$$(B^* \psi(t) - \lambda_0 \Phi_u(x_0, u_0, t), u_0(t)) =$$

$$= \max_{u \in M} (B^* \psi(t) - \lambda_0 \Phi_u(x_0, u_0, t), u)$$

a.e. on $[0,T]$.

3. Some historical remarks.

Problems of optimal control of functional differential equations with function space terminal conditions have been investigated only for the last years. Among the approaches available we cite the followings:

a) BANKS and KENT [8] considered the equation

$$\frac{d}{dt} D(x(\cdot),t) = f(x(\cdot),u(t),t) ,$$

where

$$D(x(\cdot),t) := x(t) - \int_{t_0-h}^{t} d_s\mu(t,s)x(s) .$$

The functions are taken on the interval $[t_o-h,t_1]$. This is a neutral functional differential equation with finite delay. Using Neustadt's abstract theory of extremals sufficient and necessary conditions for minimizing

$$I = \int_{t_0}^{t_1} f^o(x(t),u(t),t)\,dt$$

are derived, the latter in form of an integral maximum principle.

b) JACOBS and KAO [29] applied the abstract Lagrange multiplier rule (see LUENBERGER [36]) to nonlinear delay-equations with unconstrained L_2-controls.

c) BANKS and JACOBS [6] used an attainable set approach in $W_2^{(1)}$ for the neutral system

$$\dot{x}(t) = A_1(t)\dot{x}(t-h) + A_2(t)x(t) + A_3(t)x(t-h) + B(t)u(t)$$

with unconstrained L_2-controls and derived a result analogous to those in a) and b).

d) Recently, BANKS and MANITIUS [9] used the projection series method for linear systems

$$\dot{x}(t) = L(x_t) + Du(t)$$

with L_2-controls, constrained or unconstrained, and quasi-convex cost functional. They obtained the optimal solution as the limit of a sequence of solutions to certain finite dimensional problems.

e) DAS [18] considered a neutral difference-differential equation with constraints in the state variable and obtained a generalization of b) via the Dubovitskii-Milyutin method.

Applications of the Dubovitskii-Milyutin theory to general hereditary systems, such as Volterra integrodifferential equations or at all functional differential equations which cannot be written as difference-differential equations, seem not to occur in the literature by now. The reason may be complications concerning the definition of an "adjoint system" in connection with the transformation of the Euler equation to get the desired maximum condition.

We shall use the space C of continuous functions as phase space and L_∞-controls. An argument against L_∞-controls in the theory of optimal control for ordinary differential equations is the following: If the range M of the control function u consists of a finite number of discrete points, it is difficult to "variate" the control function in a small neighborhood of u_o, due to the L_∞-norm (see GIRSANOV [24]). Now in view of the bang-bang-principle, a finite range of M seems to be "natural" for ordinary differential equations, so this difficulty is indeed crucial. But, dealing with functional differential equations, this argument (at least) does not hold, since the bang-bang-principle is not valid (see BANKS and KENT [8]), so a finite range of the control variable in general would not allow an optimal governing of the system.

4. Definitions.

Let S be a given function space,

$$S := \{ \phi : (a,b) \to R^n \},$$

where a or b may be infinite, and x a function

$$x : I \to R^n ,$$

I some interval in R. If $t \in I$, $t-(b-a) \in I$, then we denote by x_t an element of S given by

$$x_t(\theta) := x(t + \theta - b), \quad \theta \in (a,b).$$

If $t \in I$, $t+(b-a) \in I$, then we denote by x^t an element of S given by

$$x^t(\theta) := x(t + \theta - a), \quad \theta \in (a,b).$$

To prove the main theorem we need the definition of cone and dual cone. Let X be a topological vector space and $C \subset X$. C is a cone with apex at 0, if

$$\lambda > 0, \; x \in C \Rightarrow \lambda x \in C.$$

Let C be a cone with apex at 0. Denote by X' the dual space of X (the dual space with the strong topology will be denoted by X^*). The dual cone C^* is defined by

$$C^* := \{ f \in X' : \forall x \in C : f(x) \geq 0 \}.$$

A nonzero linear functional f is called a support (or supporting functional) for a set $A \subset X$ at $x_0 \in A$, if

$$x \in A \Rightarrow f(x) \geq f(x_0).$$

5. Auxiliary material. Adjoint relations.

We shall need the following

Lemma. Let

$$L(x_t) := \int_{-\infty}^{0} G(\theta)x(t+\theta)d\theta ,$$

$$L^*(\psi^s) := \int_{-\infty}^{0} \psi(s-\theta)G(\theta)d\theta ,$$

where $x_o = 0$, $\psi^T = 0$, ψ and G of bounded variation and x continuous. Then

$$\int_0^T \psi(t)L(x_t)dt = \int_0^T L^*(\psi^s)x(s)ds$$

Proof. Substituting $s = t + \theta$ we get

$$\int_0^T \psi(t)L(x_t)dt = \int_0^T \psi(t)\int_{-\infty}^{0} G(\theta)x(t+\theta)d\theta dt =$$

$$= \int_0^T \psi(t)\int_{-\infty}^{t} G(s-t)x(s)ds$$

$$= \int_{s=0}^{T}\int_{t=s}^{T} \psi(t)G(s-t)x(s)dsdt$$

$$= \int_{s=0}^{T} [\int_{t=s}^{T} \psi(t)G(s-t)dt] x(s)ds$$

$$= \int_{s=0}^{T} [\int_{-\infty}^{0} \psi(s-\theta)G(\theta)d\theta] x(s)ds = \int_0^T L^*(\psi^s)x(s)ds .$$

6. Auxiliary material. Existence of solutions in space V.

Consider the equation

$$\psi(t) = \int_t^T (A^* \psi(s) + L^* \psi^s)\, ds + W(t) \ , \qquad (6.1)$$

$$\psi^T = 0 \ .$$

Here the $n \times n$ - Matrix $A^*(t)$ is continuous and of bounded variation, $L^* \psi^s$ is defined as in chapter 5., and $W(t)$ is of bounded variation on $[0,T]$.
We are interested in solutions of (6.1) in the space V of functions of bounded variation. The following is valid:

> Theorem. The equation (6.1) has an unique solution in the space $V[0,T]$.

We prove this theorem by applying the contractive mapping principle to the operator

$$P\varphi(t) := \int_t^T (A^* \varphi(s) + L^* \varphi^s)\, ds + W(t) \ . \qquad (6.2)$$

This operator maps the space $V[T-h,T]$ of functions of bounded variation with $\varphi(T) = 0$ into itself for small $h > 0$. The space is a Banach space with the norm

$$\|\varphi\| := \operatorname*{Var}_{T-h}^{T} \varphi(t)$$

We use the following relations:

1. $$\operatorname*{Var}_{T-h}^{T}(A^*(s)\varphi(s)) = \lim\sup\left\{\sum |A^*(t_{k+1})\varphi(t_{k+1}) - A^*(t_k)\varphi(t_k)|\right\}$$

$$\leq \lim\sup\left\{\sum |A^*(t_{k+1})\varphi(t_{k+1}) - A^*(t_k)\varphi(t_{k+1}) +\right.$$

$$\left. + A^*(t_k)\varphi(t_{k+1}) - A^*(t_k)\varphi(t_k)|\right\}$$

$$\leq \limsup \left\{ \sum \left| A^*(t_{k+1}) - A^*(t_k) \right| \left| \varphi(t_{k+1}) \right| \right\} +$$

$$+ \limsup \left\{ \sum \left| A^*(t_k) \right| \left| \varphi(t_{k+1}) - \varphi(t_k) \right| \right\}$$

$$\leq \operatorname*{Var}_{0}^{T}(A^*(t)) \cdot \sup\left\{ \varphi(t),\ T-h \leq t \leq T \right\} +$$

$$+ \sup\left\{ A^*(t),\ T-h \leq t \leq T \right\} \cdot \operatorname*{Var}_{T-h}^{T}(\varphi(s)) \leq 2M \operatorname*{Var}_{T-h}^{T}(\varphi(s)) .$$

2. $$\operatorname*{Var}_{T-h}^{T}(L^* \varphi^t) = \operatorname*{Var}_{T-h}^{T}\left(\int_{-\infty}^{0} \varphi(t-\Theta) G(\Theta) d\Theta \right) = \operatorname*{Var}_{T-h}^{T}\left(\int_{s=t} \varphi(s) G(t-s) ds \right) =$$

$$= \limsup \sum \left| \int_{t_{k+1}}^{T} \varphi(s) G(t_{k+1}-s)) ds - \int_{t_k}^{T} \varphi(s) G(t_k-s)) ds \right| =$$

$$= \limsup \sum \left| \int_{t_k}^{t_{k+1}} \varphi(s) G(t_k-s)) ds + \int_{t_{k+1}}^{T} \varphi(s) (G(t_{k+1}-s) - G(t_k-s)) ds \right|$$

$$\leq \limsup \sum \int_{t_k}^{t_{k+1}} (|\varphi(s)| \, |G(t_k-s)|) ds +$$

$$+ \limsup \sum \left| \int_{t_{k+1}}^{T} (\varphi(s) G(t_{k+1}-s) - G(t_k-s)) ds \right| .$$

Here the first summand is

$$\leq \int_{T-h}^{T} |\varphi(s)| \sup|G| \, ds \leq h \operatorname*{Var}_{T-h}^{T} \varphi \cdot M ,$$

the second summand is

$$\leq \limsup \sum \int_{T-h}^{T} |\varphi(s)| \, |G(t_{k+1}-s) - G(t_k-s)| \, ds$$

$$\leq \int_{T-h}^{T} (|\varphi(s)| \limsup \sum |G(t_{k+1}-s) - G(t_k-s)|) ds \leq$$

$$\leq h \operatorname*{Var}_{T-h}^{T}(\varphi(s))\, M.$$

So $\operatorname*{Var}_{T-h}^{T}(L^{*}\varphi^{t}) \leq 2hM\cdot\operatorname*{Var}_{T-h}^{T}(\varphi(s))$.

3. Let f be of bounded variation and set

$$F(t) := \int_{0}^{t} f(s)ds .$$

Then

$$\operatorname*{Var}_{0}^{T}(F(t)) = \lim\sup\left\{\sum\left|\int_{0}^{t_{k+1}} f(s)ds - \int_{0}^{t_k} f(s)ds\right|\right\} =$$

$$= \lim\sup\left\{\sum\left|\int_{t_k}^{t_{k+1}} f(s)ds\right|\right\} \leq \lim\sup\sum\int_{t_k}^{t_{k+1}}|f(s)|\,ds =$$

$$= \int_{0}^{T}|f(s)|\,ds \leq T\left[|f(0)| + \operatorname*{Var}_{0}^{T}(f(t))\right] .$$

By the same consideration we also derive

$$\operatorname*{Var}_{0}^{T}(F(t)) \leq T\left[|f(T)| + \operatorname*{Var}_{0}^{T}(f(t))\right] .$$

This result cannot be sharpened as one sees by $f(t) = 1$ on $[0,1)$, $f(1) = 0$.

Now from (6.2) we get

$$\|P\varphi - P\psi\| = \operatorname*{Var}_{T-h}^{T}\left(\int_{t}^{T} (A^{*}(\varphi(s) - \psi(s)) + L^{*}(\varphi^{s} - \psi^{s}))ds\right.$$

$$\leq h \operatorname*{Var}_{T-h}^{T}(A^{*}(\varphi(s) - \psi(s)) + L^{*}(\varphi^{s} - \psi^{s}))$$

$$\leq h(\operatorname*{Var}_{T-h}^{T}(A^{*}(\varphi(s) - \psi(s)) + \operatorname{Var}(L^{*}(\varphi^{s} - \psi^{s})))$$

$$\leq h(2M \cdot \operatorname*{Var}_{T-h}^{T}(\varphi(s) - \psi(s)) + 2hM \operatorname*{Var}_{T-h}^{T}(\varphi(s) - \psi(s)))$$

$$= 2hM(1 + h)\|\varphi - \psi\| \;.$$

Therefore P is a contractive operator for h sufficiently small. This implies the existence of an unique solution of equation (6.1) on $[T-h,T]$.

Now we can repeat the argument on $[T-2h,T-h]$ and continue the solution. After a finite number of steps we have proved the theorem.

Note that under our assumptions there exists a constant M such that

$$|A^*(t)| \leq M \;, \quad |G(t)| \leq M \quad \text{on } [0,T]$$

$$\operatorname*{Var}_{0}^{T}(A^*(t)) \leq M$$

$$\operatorname*{Var}_{0}^{T}(G(t)) \leq M \;.$$

7. Auxiliary material. The theorem of Dubovitskii-Milyutin.

The theorem of Dubovitskii-Milyutin provides necessary conditions for the solution of a certain extremum problem. This problem can be formulated very general so to enclose a great variety of special classical and modern extremum problems. To formulate the theorem we introduce the following notations.

Let X be a locally convex topological vector space, and $x_o \in X$. Consider the functional F defined in a neighborhood of x_o. We are interested in a local minimum of $F(x)$, where x satisfies two types of constraints:

$$(1) \quad x \in C_i, \; i = 1,2,\ldots,n$$

$$\text{where } C_i \subset X, \quad \mathring{C}_i \neq \emptyset$$

$$(2) \quad x \in C_{n+1}$$

$$\text{where } C_{n+1} \subset X, \quad \mathring{C}_{n+1} = \emptyset .$$

Usually $C_1,\ldots,C_n$ are called inequality constraints, C_{n+1} is called equality constraint.
Now let

$$C := \bigcap_{i=1}^{n+1} C_i \quad .$$

We look for a $x_o \in C$ such that

$$F(x_o) = \min_C f(x) \quad ,$$

that is, we want to determine necessary conditions that must hold for such x_o.

We have to introduce some definitions.

(a) Directions of decrease.
A $h \in X$ is called direction of decrease of the functional $F(x)$

at x_o, if there exist a neighborhood $U(h)$ and a $\alpha = \alpha(F,x_o,h) <$ < 0, such that for $0 < \varepsilon < \varepsilon_0$, $\bar{h} \in U$

$$F(x_o + \varepsilon \bar{h}) \leq F(x_o) + \varepsilon\alpha.$$

The directions of decrease generate an open cone K with apex at 0. The functional F is called regularly decreasing, if the set of its directions of decrease at the point x_o is convex.

(b) <u>Feasible directions</u>.

A $h \in X$ is called feasible direction for C_i at x_o $(i \in \{1,2,\dots,n\})$, if there exists a $U(h)$ such that for $0 < \varepsilon < \varepsilon_0$, $\bar{h} \in U$

$$x_o + \varepsilon \bar{h} \in C_i .$$

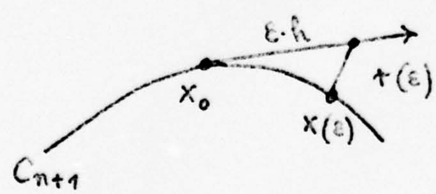
The feasible directions generate an open cone K with apex at O. We call C_i regular at x_o, if the cone of feasible directions is convex.

(c) <u>Tangent directions</u>.

A $h \in X$ is called a tangent direction to C_{n+1} at x_o, if for every neighborhood $U(0)$ there exists an $\varepsilon_0 > 0$, that for every ε, $0 < \varepsilon < \varepsilon_0$, we can find a $x(\varepsilon) \in C_{n+1}$, such that

$$\frac{1}{\varepsilon} r(\varepsilon) \in U(0),$$

where $r(\varepsilon) := x(\varepsilon) - x_o - \varepsilon h$. (In a Banach space this means $\| R(\varepsilon) \| = o(\varepsilon)$.)



(Formulation and figure in GIRSANOV [24], p. 4o could be misleading)
The tangent directions generate a cone K with apex at 0. C_{n+1} is called regular at a point x_o, if the cone of tangent directions is convex.

Now the main theorem can be given.

<u>Theorem</u>. Let $F(x)$ be regularly decreasing at x_o and denote by K_o the cone of directions of decrease. Let C_i, $i = 1,2,\ldots,n$ be regular at x_o where K_i denote the cones of feasible directions. Let C_{n+1} be regular at x_o and denote by K_{n+1} the cone of tangent directions.
Let $F(x)$ assume a local minimum on C at $x_o \in C$.
Then there exist $f_i \in K_i^* \subset X^*$, $i = 0,1,\ldots,n+1$, not all identically zero, such that

$$f_o + f_1 + \ldots + f_{n+1} = 0 . \tag{E}$$

A proof of this theorem may be found in GIRSANOV [24],p.41.

In applying the theorem to our extremum problem, the following tasks must be solved:

A. Determination of directions of decrease, feasible directions and tangent directions.
B. Construction of the dual cones K_i^*.
C. Transformation of (E) (the "Euler equation") to get the desired maximum condition.

8. Remarks on the used spaces.

The "true state" of a hereditary system governed by a FDE is an element of some function space. In examining control problems with FDEs a principle question is which function space is to be used, that is, how we have to topologize the set $F = \{x_t\}$ and which x_t we accept.

In contemporary investigations there are mainly three spaces that have proved to be useful:

a) The space C of continuous functions,

b) The Sobolew space $W_2^{(1)}$,

c) The space M_2.

Ad a): As to the domain of definition we have

$$C := \{\phi : [-r,0] \to R^n\} \quad \text{with the sup-norm,}$$

and

$$C^{\infty} := \{\phi : (-\infty,0] \to R^n\}$$

with the metric

$$d(\phi,\psi) := \sup\{\inf\{\|\phi-\psi\|_k, \tfrac{1}{2^k}\}, k = 0,1,2,\dots\}$$

where

$$\|\phi-\psi\|_k := \sup\{|\phi(t) - \psi(t)|, -k-1 \leq t \leq -k\}$$

(see ARENS [2], HALE [27])

Ad b): The space $W_2^{(1)}([-h,0],R^n)$ consits of absolutly continuous functions ϕ with $\dot{\phi} \in L_2$. This space is a Hilbert space with the scalar product

$$(\phi,\psi) := (\phi(-h),\psi(-h)) + \int_{-h}^{0} (\dot{\phi}(t),\dot{\psi}(t))\,dt .$$

(See BANKS and JACOBS [6]).

Ad c): The space M_2 is defined as

$$M_2 := R^n \times L_2([-h,0],R^n),$$

with the norm

$$\|\phi\|^2_{M_2} := |\phi^0|^2 + \|\phi^1\|_{L_2}$$

where we set $\phi = (\phi^0, \phi^1)$. (See DELFOUR [20])
The $W_2^{(1)}$ can be embedded in the M_2 in a natural way by setting

$$W^2 := \{(\phi(0),\phi) \in M_2,\ \phi \in W_2^{(1)}\} \subset M_2$$

(see MANITIUS [28])

In applying the method of Dubovitskii-Milyutin to Volterra integro-differential equations, two points have to be watched. The functions $x(t)$ are defined on $(-\infty,T]$. That is, from the preceding spaces only the C^∞ can be considered as an appropriate space. But this space is (only) a metric space, not a Banach space.

One crucial point in the application of the method is the determination of the cone of tangent directions to the equality constraint, that is to the solutions of the equation. The determination of this cone works via the theorem of Lyusternik, which assumes Banach spaces.

Therefore we have to introduce a new space, where the following considerations take place:

1. It should be a Banach space, so that the theorem of Lyusternik can be applied.

2. It should contain the continuous functions on $(-\infty,T]$, since usually system states vary continuously.

These are the motivations for introducing the space

$$\bar{C}^\infty(-\infty,\alpha] := \{\phi : (-\infty,\alpha] \to R^n,\ \phi \text{ bounded and continuous}\}$$

with $\|\phi\| = \sup\{|\phi(t)|,\ -\infty < t \leq \alpha\}$.

9. Existence and uniqueness.

In this chapter we show, that the equation

$$(V)\quad \begin{cases} \dot{x}(t) = A(t)x(t) + \int_{-\infty}^{t} F(t-\tau)x(\tau)d\tau + B(t)u(t) \\ \text{a.e.}[0,T] \\ \\ x_0 = \phi_0 \in \tilde{C}^{\infty}(-\infty,0] \end{cases}$$

$A(t)$, $B(t)$ continuous on $[0,T]$.

$F(s)$ of bounded variation, $\int_{-\infty}^{T} |F(s)|ds < \infty$

$u \in L_{\infty}[0,T]$

has a unique solution on $[0,T]$.
For this proof we use a method introduced by MORGENSTERN [41],
in this connection first used by STETTNER [50]. The method is based
on the following Lemma by CHU and DIAZ [10]:

<u>Lemma</u>. Let S be an arbitrary set and A,K mappings,

$$A: S \to S\ , \quad K: S \to S\ .$$

Assume there exists a right inverse to K, that is, a K_r^{-1},
such that $KK_r^{-1} = I$.
Then A has a fixed point in S iff $K_r^{-1}AK$ has a fixed point
in S.

The proof of this lemma is almost self-evident.
To prove the unique existence of a solution of (V) in the
space of bounded continuous functions on $(-\infty,T]$, we introduce
the operators A and K_λ by

$$Ax(t) := \begin{cases} \phi_0(t) & -\infty < t \leq 0 \\ \phi_0(0) + \int_0^t (Ax(s) + \int_{-\infty}^{s} F(s-\tau)x(\tau)d\tau + Bu(s))ds\ , \end{cases}$$

$0 \leq t \leq T$

$$K_\lambda x(t) := \begin{cases} x(t) & -\infty < t \leq 0 \\ e^{\lambda t} x(t) & 0 \leq t \leq T . \end{cases}$$

K_λ has an inverse operator, namely K_λ^{-1}:

$$K_\lambda^{-1} x(t) = \begin{cases} x(t) & -\infty < t \leq 0 \\ e^{-\lambda t} x(t) & 0 \leq t \leq T. \end{cases}$$

Now consider the operator $K_\lambda^{-1} A K_\lambda$. For two elements x,y of $\bar{C}^\infty(-\infty, T]$ we get

$$\|K^{-1}AK\, x - K^{-1}AK\, y\| \leq$$

$$\leq \sup \left\{ e^{-\lambda t} \left| (AK_\lambda x - AK_\lambda y)(t) \right| \; ; \; 0 \leq t \leq T \right\}$$

$$\leq \sup \left\{ e^{-\lambda t} \int_0^t \left[e^{\lambda s} M_1 \|x - y\| + \|x - y\| V \int_0^s e^{\lambda \tau} d\tau \right] ds \, , \quad 0 \leq t \leq T \right\}.$$

Here

$$\int_0^t e^{\lambda s} M_1 \|x - y\| \leq M_1 \lambda^{-1} \|x - y\| e^{\lambda t} \, ,$$

$$\int_0^t \int_0^s e^{\lambda \tau} d\tau \, ds \leq e^{\lambda t} \lambda^{-2},$$

We used

$$|A(t)| \leq M_1, \quad |F(s)| \leq V \, .$$

Now the sup is

$$\leq \sup \left\{ M_1 \lambda^{-1} \|x - y\| + V \cdot \lambda^{-2} \|x - y\| \right\}$$

$$\leq M_2 \lambda^{-1} \|x - y\|$$

where M_2 can be chosen independent of λ, x,y.
So we see for $\lambda > M_2$ this is a contraction. This proves $K_\lambda^{-1} A K_\lambda$ has a fixed point in $\tilde{C}^\infty(-\infty, T]$, which is unique determined. Applying the lemma we derive the existence of a (not necessary the same!) unique fixed point of A in $\tilde{C}^\infty(-\infty, T]$, which gives the desired solution.

1o. The cone of tangent directions.

Consider the equation

$$\dot{x}(t) = Ax(t) + Lx_t + Bu(t), \qquad (10.1)$$

where

$$Lx_t := \int_{-\infty}^{t} F(t-\tau)x(\tau)d\tau \quad ,$$

with the boundary conditions

$$x_0 = \phi_0 \in \bar{C}^{\infty}$$

$$x_T = \phi \in C[T-r,T], \qquad (T-r > 0),$$

under the hypotheses of chapter 9. This equation defines a certain set $C_{n+1} = \{(x,u)\}$ in the space $\bar{C}^{\infty} \times L_{\infty}$. We want to characterize the cone of tangent directions to the set C_{n+1}.
Introduce the operator

$$P: \bar{C}^{\infty} \times L_{\infty} \to \bar{C}^{\infty} \times C[T-r,T]$$

by

$$P(x,u) := \begin{cases} x(t) - \phi_0(t) \ , & -\infty < t \leq 0 \\ x(t) - \phi_0(0) - \int_0^t (Ax(s)+Lx_s+Bu(s))ds, & 0 \leq t \leq T \ ; \end{cases}$$

$$; \ x_T - \phi \}$$

Then the solutions of equation (1o.1) are the zeros of the operator P, that is, they satisfy

$$P(x,u) = 0.$$

To find the cone of targent directions to the solutions of the last equation, we use the following theorem of Lysternik (see LYUSTERNIK and SOBOLEW [35]) :

Theorem. Let X,Y be Banach spaces, P an Operator

$$P: X \to Y$$

vanishing for $x_o \in X$. Assume the Frechet-derivative $P'(x)$ exists and is continuous in a neighborhood of x_o. Let $P'(x_o)$ be surjective.
Denote by K the cone of tangent directions in x_o to the set $Q = \{x : P(x) = 0\}$.
Then

$$K = \{h : P'(x_o)h = 0\}.$$

To apply the theorem of Lyusternik to our problem we first show that the operator P has a continuous Frechet-derivative.
We see

$$P(x + \bar{x}, u + \bar{u}) - P(x,u) =$$

$$= \left\{ \begin{cases} \bar{x}(t) , & -\infty < t \leq 0 \\ \bar{x}(t) - \int_0^t (A\bar{x}(s) + L\bar{x}_s + B\bar{u}(s))ds, & 0 \leq t \leq T \end{cases} ; x_T \right\}$$

Since the right side id a linear continuous operator, we get

$$P'(x,u)(x,u) =$$

$$= \left\{ \begin{cases} \bar{x}(t) , & -\infty < t \leq 0 \\ \bar{x}(t) - \int_0^t A\bar{x}(s) + L\bar{x}_s + B\bar{u}(s))ds, & 0 \leq t \leq T \end{cases} ; \bar{x}_T \right\}$$

This follows at once from the definition of the Frechet-derivative,

namely the validity of

$$P(x_o + h) - P(x_o) = P'(x_o)h + r(x_o,h),$$

where $\|r\| = o(\|h\|)$. In our case $r \equiv 0$. Note, that P is not linear!
The continuity of P' follows by known arguments (see LUENBERGER [36], p.193).
Now we want to show that $P'(x,u)$ is surjective. That is, for arbitrary

$$((a(t),b_T) \in \bar{C}^{\infty}(-\infty,T] \times C[T-r,T] .$$

we must solve

$$(S) \begin{cases} x(t) = a(t) & -\infty < t \leq 0 \\ x(t) = a(t) + \int_0^t (Ax(s) + Lx_s + Bu(s))ds & 0 \leq t \leq T \\ x_T = b_T \end{cases}$$

This system has a solution, if the condition of complete controllability is valid. Therefore we assume

(<u>CCC</u>): The System $(S)'$ is completely controllable.

Then we can apply the theorem of Lyusternik and get the following result:

The cone of tangent directions to the set C_{n+1} is given by

$$K := \Big\{ \bar{x},\bar{u}: \bar{x}_o = 0, \bar{x}(t) = \int_0^t (A\bar{x}(s) + L\bar{x}_s + B\bar{u}(s))ds,$$
$$0 \leq t \leq T, \bar{x}_T = 0 \Big\} \subset \bar{C}^{\infty}(-\infty,T] \times L_{\infty}[0,T]$$

The condition of complete controllability means the following:
Consider the system $(S)'$

$$(S)' \begin{cases} \dot{y}(t) = Ay(t) + Ly_t + bu(t) & 0 \leq t \leq T \\ y(t) = 0, \quad -\infty < t \leq 0 \\ u \in L_\infty[0,T] \end{cases}$$

As shown in chapter 9., this system has a solution on $[0,T]$. Denote by F the set of all $y_T \in C[T-r,T]$, where $y(t)$ is a solution of $(S)'$, as u ranges over all of $L_\infty[0,T]$.
(S)' is called completely controllable if

$$F = C[T-r,T] .$$

Now, if (S)' is completely controllable, (S) has a solution. To prove this, chose z(t) such that

$$z(t) = a(t) , \quad -\infty < t \leq 0$$

$$z(t) = a(t) + \int_0^t (Az(s) + Lz_s)ds \qquad 0 \leq t \leq T ,$$

(this has a solution by chapter 9.), and then chose $y(t)$ such that

$$y(t) = 0 , \quad -\infty < t \leq 0$$

$$y(t) = \int_0^t (Ay(s) + Ly_s + b\tilde{u}(s))ds \qquad 0 \leq t \leq T$$

$$y_t = b_T - z_T$$

(this has a solution by the condition of complete controllability).
Now,

$$x(t) := y(t) + z(t)$$
$$u(t) := \tilde{u}(t)$$

is a solution of (S).

11. The dual cone.

We want to determine the dual cone to the cone of tangent directions derived in chapter 1o. Since K is a subspace of

$$X := \bar{C}^{\infty}(-\infty,T] \times L_{\infty}[0,T] \quad ,$$

we get (see GIRSANOV [24], p.69)

$$K^{*} = \{ f \in X^{*} : f(\bar{x},\bar{u}) = 0 \quad \text{for} \quad (\bar{x},\bar{u}) \in K \} \; .$$

Let us characterize the functionals in K^*. To do so we use a lemma about adjoint operators (see LUENBERGER [36], p.156):

Lemma. Let X,Y be Banach spaces and T a linear operator, $T: X \to Y$, bounded. Assume that the range of T be closed. Denote with T^* the adjoint operator of T, with $\mathcal{R}(T^*)$ the range of this operator, with $\mathcal{N}(T)$ the nullspace of T and with $S^{\perp}$, $S \subset X$, the "orthogonal complement", that is

$$S^{\perp} := \{ x^* \in X^* : x^*(x) = 0 \quad \text{for all} \quad x \in S \} \; .$$

Then

$$\mathcal{R}(T^*) = [\mathcal{N}(T)]^{\perp} \; .$$

Now define an operator

$$T : \bar{C}^{\infty}(-\infty,T] \times L_{\infty}(0,T) \to \bar{C}^{\infty}(-\infty,T] \times C[T-r,T]$$

by

$$T(\bar{x},\bar{u}) :=$$

$$= \left\{ \begin{cases} \bar{x}(t) & -\infty < t \le 0 \\ \bar{x}(t) - \int_0^t (A\bar{x}(s) + L\bar{x}_s + Bu(\bar{s}))ds, & 0 \le t \le T \end{cases} \; ; \; \bar{x}_T \right\}$$

(actually this is the operator $P'(x,u)$ derived in chapter 1o.), and denote elements of $\bar{C}^{\infty}(-\infty,T] \times L_{\infty}[0,T]$ shortly by x,y.

Then

$$K^* = \{f \in X^* : f(x) = 0 \quad \text{for} \quad Tx = 0\} \quad ,$$

that is, x is "orthogonal" to f, if $x \in \mathcal{N}(T)$.
Now $\mathcal{R}(T)$ is closed, since the system is completely controllable. Therefore we can apply the previous theorem with the following result:
f must be in the range of T^*, that is, there exists a y,

$$y \in [\bar{C}^{\infty}(-\infty,T] \times C[T-r,T]]^* \quad =: D$$

such that $f = T^* y$. Hence

$$f(x) = (T^* y)(x) = y(Tx) \quad .$$

Now the elements of the dual space D have the form (d,θ), where θ is a function of bounded variation on $[T-r,T]$ (to be more precise, the second coordinates form a space isomorphic to the space of functions of bounded variation). The form of the first coordinate need not be known, as soon will be seen.
If we choose

$$\bar{x}_0(t) = 0 \qquad -\infty \leq t \leq 0,$$

$$\bar{x}(t) = \int_0^t (Ax(s) + Lx_s + Bu(s))ds \quad , \quad 0 \leq t \leq T \quad ,$$

then the first summand in $y(Tx)$ vanishes and we get

$$f(x) = \langle x_T, \theta \rangle \, ,$$

where $\langle \cdot , \cdot \rangle$ denotes the "pairing" of elements of $C[T-r,T]$ with elements of its dual space, that is,

$$f(x) = \int_{T-r}^{T} x(s) d\theta(s) \quad ,$$

the latter being a Stieltjes-integral.

12. The Euler Equation.

Now consider our control problem (CP). Let x_o, u_o be a solution of this problem. We shall establish the Euler equation by applying the theorem of Dubovitskii-Milyutin. To do this, we have to determine the cone of decrease and his dual cone for the functional

$$F(x,u) = \int_0^T \phi(x(t),u(t),t)\,dt .$$

Using a known result concerning functionals of this type (see for instance GIRSANOV [24], p.56, Example 7.7 and p.69, theorem 10.2) we get:

If $K_o \neq \emptyset$, then for any $f_o \in K_o^*$,

$$f_o(\bar{x},\bar{u}) = -\lambda_o \int_0^T (\phi_x(x_o,u_o,t),\bar{x}) + (\phi_u(x_o,u_o,t),u))dt,$$

where $\lambda_o \geq 0$.

Next we analyze the constraint

$$u(t) \in M \qquad \text{for } 0 \leq t \leq T \text{ a.e.}$$

Denote by K_1 the cone of feasible directions on the set Q_1 at x_o, u_o, where

$$Q_1 = C \times Q_1 ,$$

$Q_1' \subset L_\infty$ the set of functions $u(t)$ with values in M a.e. Repeating an argument of Girsanov [24], p.86, b), we get:

If $f_1 \in K_1^*$, then $f_1 = (0, f_1')$, where $f_1' \in L_\infty^*$ is a support to Q_1' at u_o.

Now we apply the theorem of Dubovitskii-Milyutin to (CP) and derive:

There exist functionals

$$f_o, f_1, f_2 \in [\tilde{C}^{\infty}(-\infty, T] \times L_{\infty}(0,T)]^* ,$$

not all zero, such that

$$f_o(\tilde{x},\tilde{u}) + f_1(\tilde{x},\tilde{u}) + f_2(\tilde{x},\tilde{u}) = 0 , \qquad (12.1)$$

where f_o, f_1 just were defined, and $f_2 \in K^*$ (see chapter 11.), that is,

$$f_2(x) = \int_{T-r}^{T} x(s)d\theta(s)$$

for

$$\bar{x}_o(t) = 0 \qquad -\infty < t \leq 0$$

$$\bar{x}(t) = \int_0^t (A\bar{x}(s) + L\bar{x}_s + B\bar{u}(s))ds, \quad 0 \leq t \leq T \qquad (12.2)$$

Then (12.1) becomes

$$f_1'(u) = \lambda_o \int_0^T ((\phi_x(x_o,u_o,t),x) + (\phi_u(x_o,u_o,t),u))dt -$$

$$- \int_{T-r}^{T} \bar{x}(s)d\theta(s) \qquad (12.3)$$

In the following we transform the right side of (12.3) by using the function ψ defined by

$$\psi(t) = \int_t^T (A^*\psi + L^*\psi^s - \lambda_o\phi_x)ds + \theta(T) - \theta(t),$$

$$0 \leq t \leq T$$

$$\psi^T = 0, \text{ (especially } \psi(T) = 0) . \qquad (12.4)$$

Here we assume $\theta(T-r) = 0$ (that is, θ is <u>normalized</u> with respect to the Stieltjes-integral in (12.3), see PFLAUMANN-UNGER [40]) and

$$\theta(t) = 0 \quad \text{for} \quad 0 \leq t < T - r.$$

In chapter 6. we proved the existence of solution to this equation.

Now we first want to show that

$$\lambda_0\int_0^T(\phi_x\,\bar{x})dt - \int_{T-r}^T \bar{x}d\theta = -\int_0^T(B^*\psi,\bar{u})ds \qquad (12.5)$$

The right side of this equation is

$$-\int_0^T(B^*\psi,\bar{u})dt = -\int_0^T(\psi,B\bar{u})dt$$

$$= -\int_0^T(\psi,\dot{\bar{x}} - A\bar{x} - L\bar{x}_t)dt \qquad (12.6)$$

We shall show

$$\int_{T-r}^T((\psi,\dot{\bar{x}} - A\bar{x} - L\bar{x}_t) + (\lambda_0\phi_x,x))dt =$$

$$= \int_{T-r}^T \bar{x}d\theta - \psi(T-r)x(T-r)\ , \qquad (12.7)$$

$$\int_{0,T-T}^{T-r}((\psi,\dot{\bar{x}} - A\bar{x} - L\bar{x}_t) + (\lambda_0\phi_x,x))dt =$$

$$= \psi(T-r)x(T-r)\ . \qquad (12.8)$$

Addition of (12.7) and (12.8) using (12.6) proves (12.5). Now with (12.4) we get

$$\int_{T-r}^T((\psi,\dot{\bar{x}} - A\bar{x} - L\bar{x}_t) + (\lambda_0\phi_x,x))dt =$$

$$= \int_{T-r}^T(\int_t^T(A^*\psi + L^*\psi^s - \lambda_0\phi_x)ds + \theta(T) - (\theta(t)),\dot{\bar{x}}) -$$

$$- \psi A\bar{x} - \psi L\bar{x}_t + \lambda_0\phi_x\,\bar{x})dt =$$

$$= \int_{T-r}^T\dot{\bar{x}}\int_t^T(A^*\psi + L^*\psi^s - \lambda_0\phi_x)dsdt - \int_{T-r}^T\theta\dot{\bar{x}}dt + \theta(T)\int_{T-r}^T\dot{\bar{x}}dt -$$

$$- \int_{T-r}^T(\psi A\bar{x} - \psi L\bar{x}_t + \lambda_0\phi_x\,\bar{x})dt =$$

$$= \Big[\int_t^T (A^*\psi + L^*\psi^s - \lambda_0\phi_x)ds, \bar{x}(t)\Big]_{T-r}^{T} -$$

$$- \int_{T-r}^{T} -(A^*\psi + L^*\psi^t - \lambda_0\phi_x, \bar{x})dt - \int_{T-r}^{T} \theta\dot{\bar{x}}dt +$$

$$+ \theta(T)x(T) - \theta(T)x(T-r) -$$

$$- \int_{T-r}^{T} (\psi A\bar{x} - \psi L\bar{x}_t + \lambda_0\phi_x\bar{x})dt =$$

$$= -\Big(\int_{T-r}^{T} (A^*\psi + L^*\psi^t - \lambda_0\phi_x)dt, \bar{x}(T-r)\Big) - \int_{T-r}^{T} \theta\dot{\bar{x}}dt +$$

$$+ \theta(T)\bar{x}(T) - \theta(T)\bar{x}(T-r) =$$

$$= -(\psi(T-r) - \theta(T) + \theta(T-r), \bar{x}(T-r)) - \theta\bar{x}\Big|_{T-r}^{T} +$$

$$+ \int_{T-r}^{T} \bar{x}d\theta + \theta(T)\bar{x}(T) - \theta(T)\bar{x}(T-r) =$$

$$= \int_{T-r}^{T} \bar{x}d\theta - \psi(T-r)x(T-r) ,$$

where we used a well knoen theorem about Stieltjes-integrals, namely

$$\int_{\alpha}^{\beta} \bar{x}d\theta = \bar{x}\theta\Big|_{\alpha}^{\beta} - \int_{\alpha}^{\beta} \theta d\bar{x} ,$$

if one of the integrals exists (see Natanson [43], p. 257) and

$$\int \theta d\bar{x} = \int \theta\dot{\bar{x}}dt .$$

From the left side of (12.8) we get

$$\int_0^{T-r} ((\psi, \dot{\bar{x}} - A\bar{x} - L\bar{x}_t) + \lambda_0\phi_x\bar{x})dt =$$

$$= \int_0^{T-r} \Big(\int_t^T (A^*\psi + L^*\psi^s - \lambda_0\phi_x)ds + \theta(T), \dot{\bar{x}}\Big) -$$

$$- \int_0^{T-r} (\psi, A\bar{x} + \psi L\bar{x}_t - \lambda_0 \phi_x \bar{x}) dt =$$

$$= \left(\int_t^T (A^* \psi + L^* \psi^s - \lambda_0 \phi_x) ds, \bar{x}(t) \right) \Big|_0^{T-r} -$$

$$- \int_0^{T-r} -(A^* \psi + L^* \psi^t - \lambda_0 \phi_x, \bar{x}) dt + \int_0^{T-r} \Theta(T) \dot{\bar{x}} dt -$$

$$- \int_0^{T-r} (\psi, A\bar{x}) + \psi L\bar{x}_t - \lambda_0 \phi_x \bar{x}) \, dt =$$

(since $\bar{x}(0) = 0$)

$$= \left(\int_{T-r}^T (A^* \psi + L^* \psi^t - \lambda_0 \phi_x) dt, \bar{x}(T-r) \right) + \Theta(T) \bar{x}(T-r) =$$

$$= (\psi(T-r) - \Theta(T) + \Theta(T-r), \bar{x}(T-r)) + \Theta(T) \bar{x}(T-r) =$$

$$= \psi(T-r) \bar{x}(T-r) \; .$$

This proves (12.5). Now (12.3) becomes

$$f_1'(\bar{u}) = \int_0^T (-B^* \psi(t) + \lambda_0 \phi_u(x_o, u_o, t), \bar{u}(t)) dt \qquad (12.8)$$

with arbitrary $\bar{u}$, where $f_1'(\bar{u})$ is a support to Q_1' at u_o.
We use now the following lemma (see Girsanov [24], p.76, example 10.5):

<u>Lemma</u>. If the linear functional

$$f_1'(\bar{u}) = \int_0^T (a(t), \bar{u}(t)) dt \; , \qquad a(t) \in L_1(0,T)$$

is a support to Q_1' at u_o, then $u \in M$ implies

$$(a(t), u - u_o) \geq 0 \qquad \text{a.e.} [0,T] \; .$$

With this lemma, we get from (12.8):
$u \in M$ implies

$$(-B^{*}\psi(t) + \lambda_0\phi_u(x_o,u_o,t),u - u_o) \geq 0 \qquad \text{a.e.}$$

or

$$(B^{*}\psi(t) - \lambda_0\phi_u(x_o,u_o,t),u_o(t)) =$$

$$= \max_{u\in M} (B^{*}\psi(t) - \lambda_0\phi_u(x_o,u_o,t),u) \qquad \text{a.e. } [0,T] \; ,$$

what we wanted to show.

We still have to prove that the case $\lambda_0 = 0$ and $\psi(t) \equiv 0$ is impossible. Indeed otherwise we had

$$f_o(\bar{x},\bar{u}) = -\lambda_0\int_0^T(\phi_x,\bar{x}) + (\phi_u,\bar{u})\, dt \equiv 0 \; ,$$

which, using (12.4), would imply $\theta(t) = 0$, $f_1'(u) = 0$, and the first part of $f_2(\bar{x},\bar{u})$ had to vanish (in addition to $\int\bar{x}d\theta = 0$).
So we would have a contradiction to

$$\text{nontrivial}(f_o,f_1,f_2) \; .$$

Now we have to discuss the assumption $K_o \neq \theta$.
If $K_o = \theta$, then we get

$$\int_0^T((\phi_x,\bar{x}) + (\phi_u,\bar{u})dt = 0$$

for all $\bar{x},\bar{u}$. We set $\lambda_0 = 1$, $\theta \equiv 0$, and using an argument as before, we get

$$\int_0^T(\phi_x,\bar{x})dt = -\int_0^T (B^{*}\psi,\bar{u})dt.$$

So

$$\int_0^T(-B^{*}\psi + \phi_u,\bar{u})dt = 0 \quad \text{for all } \bar{u}.$$

Hence, $-B^{*}\psi + \phi_u = 0$ a.e. $[0,T]$. That is, the maximum condition is valid again.

13. Remarks.

The preceding result is not unexpected. Indeed it has the usual form of such maximum principles. It is similar to the results cited in chapter 3. or to the "classical" results for ordinary differential equations (see LEE and MARKUS [34]). Though maximum principles of this kind are satisfactory from a theoretical point of view, for the explicite computation of the optimal control function only half work is done. The next step must be the numerical evaluation of the condition. This evaluation is possible for special cases of ordinary differential equations and shall be investigated for Volterra integro-differential equations in the next part of this work.

Literature.

[1] Akcasu, A.Z., G.S. Lellouche and L.M. Shotkin:
Mathematical Methods in Nuclear Reactor Dynamics.
Academic Press, New York, 1971.

[2] Arens, R.F.:
A topology for spaces of transformations.
Ann. of Math. 47, 48o - 495(1946)

[3] Banks, H.T.:
Necessary conditions for control problems with variable time lags.
SIAM J. Control 6, 9 - 47(1968)

[4] Banks, H.T..
Variational problems involving functional differential equations.
SIAM J. Control 7, 1 - 17(1969)

[5] Banks, H.T., and M.Q. Jacobs:
The optimization of trajectories of linear functional differential equations.
SIAM J. Control 8, 461 - 488(197o)

[6] Banks, H.T., and M.Q. Jacobs:
An attainable set approach to optimal control of functional differential equations with function space terminal conditions.
J. differential Eqs. 13, 127 - 144(1973)

[7] Banks, H.T., M.Q. Jacobs and C.E. Langenhop:
Function space controllability for linear functional differential equations.
Proc.Conf.Diff.Games Control Theory, Univ.Rhode Island,1973.

[8] Banks, H.T., and G.A. Kent:
Control of functional differential equations of retarded and neutral type to target sets in function space.
SIAM J. Control 1o, 567 - 594(1972)

[9] Banks, H.T., and A.Manitius:
Projection series for retarded functional differential equations with applications to optimal control problems.
J. differential Eqs.

[1o] Chu, S.C., and J.B. Diaz:
On "in the large" application of the contraction principle.
(in: Differential equations and dynamical systems.
J.K. Hale and J.P. LaSalle, eds.)
Academic Press, New York, 1967.

[11] Chyung, D.H., and E.B. Lee:
Linear optimal systems with time delays.
SIAM J. Control 4, 568 - 575(1966)

[12] Coleman, B.D., and V.J. Mizel:
On the stability of solutions of functional differential equations.
Arch. rational Mech. Analysis 3o, 173 - 196(1968)

[13] Coleman, B.D., and V.J. Mizel:
On the general theory of fading memory.
Arch. rational Mech. Analysis 29, 18 - 31(1968)

[14] Connor, M.A.:
Time optimal control of neutral systems with control integral constraint.
J. math. Analysis Appl. 45, 91 - 95(1974)

[15] Cooke, K.L., and D.W. Krumme:
Differential-difference equations and nonlinear initial-boundary value problems for linear hyperbolic partial differential equations.
J. math. Analysis Appl. 24, 372 - 387(1968)

[16] Cooke, K.L., and J. Yorke:
Some equations modelling growth processes and gonorrhea epidemics.
(in: Ordinary Differential Equations. L. Weiss, ed.)
Academic Press, New York, 1972.

[17] Cooke, K.L., N. DiStefano and B. Kashef:
On a class of hereditary processes in biomechanics.
USC Technical Report No. 72-5 (1972)

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[18] Das, P.C.:
Application of Dubovitskii-Milyutin formalism to optimal settling problems with constraints.
Private communication, 1975.

[19] Das, P.C., and R.R. Sharma:
On optimal control for measure delay-differential equations.
SIAM J. Control 9, 43 - 61(1971)

[20] Delfour, M.C.:
State theory of linear hereditary differential systems.
Report CRM-395, Centre de Recherches Mathematiques, Universite de Montreal, 1974

[21] DiPasquantonio, F., and F. Kappel:
A stability criterion for the kinetic reactor equations with linear feedback.
Energia nucleare 19, 162 - 175(1972)

[22] DiStefano, N.:
A Volterra integral equation in the stability of some linear hereditary phenomena.
J. math. Analysis Appl. 23, 365 - 383(1968)

[23] Gamkrelidze, R.V., and G.L. Kharatishvili:
Extremal problems in linear topological spaces, I.
Math. Systems Theory 1, 229 - 256(1967)

[24] Girsanov, I.V.:
Lectures on Mathematical Theory of Extremum Problems.
Springer, Berlin, 1972.

[25] Grossman, S.I., and R.K. Miller:
Nonlinear Volterra integrodifferential systems with L^1 kernels.
J. differential Eqs. 13, 551 - 566(1973)

[26] Halanay, A., and Vl. Rasvan:
Frequency domain criteria for nuclear reactor stability.
Rev. Roumaine Sci. Tech. Electrotechn. et Energ. 19, 367 - 378(1974)

[27] Hale, J.K.:
Sufficient conditions for stability and instability of autonomuous functional-differential equations.
J. differential Eqs. 1, 452 - 482(1965)

[28] Jacobs, M.Q.:
Attainable sets in systems with unbounded controls.
J. differential Eqs. 4, 4o8 - 423(1968)

[29] Jacobs, M.Q., and T.J. Kao:
An optimum settling problem for linear time lag systems.
J. math. Analysis Appl. 4o, 687 - 7o7(1972)

[3o] Keller, K.:
Qualitative behavior of integrodifferential systems with applications in reactor dynamics.
J. differential Eqs. 16, 574 - 6o2(1974)

[31] Kent, G.A.:
Optimal control for functional differential equations of neutral type.
Ph.D. Thesis, Brown University, 1971.

[32] Kharatishvili, G.L.:
The maximum principle in the theory of optimal processes with time lag.
Doklady Akad. Nauk SSSR 136, 39 - 42(1961)

[33] Kikuchi, N.:
On control problems for functional-differential equations.
Funkcialaj Ekvacioj, Ser. internac. 14, 1 - 23(1971)

[34] Lee, E.B., and L. Marcus:
Foundations of Optimal Control Theory.
J. Wiley, New York, 1967.

[35] Ljusternik, L.A., and W.I. Sobolew:
Elements of Functional Analysis.
Nauka, Moskau, 1965.

[36] Luenberger, D.:
Optimization by Vector Space Methods.
J. Wiley, New York, 1969.

[37] MacCamy, R.C.:
Functional differential equations in continuum mechanics.
Proc.Park City Symp. Delay functional Diff.Eqs.Appl., Academic Press, New York, 1972.

[38] Manitius, A.:
Optimal control of hereditary systems.
Lecture Notes, December 1974.

[39] Manitius, A.:
Mathematical models of hereditary systems.
Report CRM-462, Centre de Recherches Mathematiques,
Universite de Montreal, 1974.

[4o] Miller, R.K.:
Nonlinear Volterra Integral Equations.
W.A. Benjamin, Menlo Prks, 1971.

[41] Mohler, R.R., and C.N. Shen:
Optimal Control of Nuclear Reactors.
Academic Press, New York, 197o.

[42] Morgenstern, D.:
Beiträge zur nichtlinearen Funktionalanalysis.
Diss. TU Berlin, 1952.

[43] Natanson, I.P.:
Theorie der Funktionen einer reellen Veränderlichen.
Akademie-Verlag, Berlin, 1969.

[44] Neustadt, L.W.:
An abstract variational theory with applications to a
broad class of optimization problems, I. General theory.
SIAM J. Control <u>4</u>, 5o5 - 527(1966)

[45] Neustadt, L.W.:
An abstract variational theory with applications to a
broad class of optimization problems, II. Applications.
SIAM J. Control <u>5</u>, 9o - 137(1967)

[46] Neustadt, L.W.:
A general theory of extremals.
J. Comput. System Sci. <u>3</u>, 57 - 92(1969)

[47] Neustadt, L.W.:
Optimization: A Theory of Necessary Conditions.
To appear.

[48] Pflaumann, E., and H. Unger:
Funktionalanalysis I.
Bibliographisches Institut, Mannheim, 1968.

[49] Rasvan, Vl.:
Absolute stability of a class of control systems described by coupled delay-differential and difference equations.
Rev. Roumaine Sci. Techn. Electrotechn. et Energ. 18, 329 - 346(1973)

[50] Stettner, H.:
Nichtlokale Existenz der Lösungen von Funktional-Differentialgleichungen.
Periodica Math. Hungarica 5, 33 - 39(1974)

[51] Volterra, E.:
Vibrations of elastic systems having hereditary characteristics.
J. appl. Mech. 17, 363 - 371(1950)

[52] Volterra, E.:
On elastic continua with hereditary characteristics.
J. appl. Mech. 18, 273 - 279(1951)

[53] Weaver, L.E.:
System Analysis of Nuclear Reactor Dynamics.
Rowann and Littlefield, New York, 1963.